2.2

27

(a) $\dfrac{dy}{dx} = e^{x^2}$, $y(0)=0$.

$$\Longleftrightarrow \int_0^x dy = \int_0^x e^{t^2}\,dt$$

$$\Longleftrightarrow y(x) - \cancel{y(0)}^{0} = \int_0^x e^{t^2}\,dt$$

$$\Longleftrightarrow \boxed{y(x) = \int_0^x e^{t^2}\,dt}$$

(c) $\dfrac{dy}{dx} = \sqrt{1+\sin x}\,(1+y^2)$, $y(0)=1$

$$\Longrightarrow \int_0^x \frac{1}{1+y^2}\,dy = \int_0^x \sqrt{1+\sin t}\,dt$$

$$\Longleftrightarrow \arctan(y(x)) - \arctan(y(0)) = \int_0^x \sqrt{1+\sin t}\,dt$$

$$\Longleftrightarrow \arctan(y(x)) - \cancel{\frac{\pi}{4}} = \int_0^x \sqrt{1+\sin t}\,dt \quad + \frac{\pi}{4}$$

$$+ \cancel{\frac{\pi}{4}}$$

$$\Longrightarrow \boxed{y(x) = \tan\left(\frac{\pi}{4} + \int_0^x \sqrt{1+\sin t}\,dt\right)}$$

$\boxed{2.3}$

(17) $\dfrac{dy}{dx} - \dfrac{y}{x} = xe^x$, $y(1) = e-1.$

$\Leftrightarrow \quad \dfrac{dy}{dx} + \left(-\dfrac{1}{x}\right)y = xe^x$

Now we find integrating factor

$$\mu(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln|x|} = \frac{1}{e^{\ln|x|}} = \frac{1}{x}.$$

Now we know that

$$\mu(x)\,y = \int \mu(x)\,xe^x\,dx$$
$$= \int e^x\,dx$$
$$= e^x + C.$$

Divide by $\mu(x)$:

$$y = \frac{e^x + C}{1/x} = xe^x + Cx.$$

So

$$e-1 = y(1) = e + C \Rightarrow C = -1$$

$$\boxed{y = xe^x - x}$$

2.3

(21) $\cos x \dfrac{dy}{dx} + y \sin x = 2x\cos^2 x$, $y\left(\frac{\pi}{4}\right) = \dfrac{-15\sqrt{2}\,\pi^2}{32}$.

$$\Longleftrightarrow \quad \frac{dy}{dx} + (\tan x)\, y = 2x\cos x \quad (*)$$

$$\mu(x) = e^{\int \tan x \, dx} = e^{\ln|\sec x|} = \sec x.$$

Multiply $(*)$ by $\mu(x)$ and integrate

$$\mu(x)\, y = \int \mu(x)\, 2x\cos x \, dx$$
$$= \int 2x \, dx$$
$$= x^2 + C.$$

So $y = (x^2 + C)\cos x$. Using initial conditions

You get $C = -\pi^2$. So

$$\boxed{y = (x^2 - \pi^2)\cos x.}$$

2.4

(15) $\cos\theta \, dr - (r\sin\theta - e^{\theta})\, d\theta = 0$

Let $M = \cos\theta$ and $N = e^{\theta} - r\sin\theta$

$\frac{\partial M}{\partial \theta} = -\sin\theta$ and $\frac{\partial N}{\partial r} = -\sin\theta$.

The equation is exact.

$$F(r,\theta) = \int M \, dr + g(\theta)$$

$$= \int \cos\theta \, dr + g(\theta)$$

$$= r\cos\theta + g(\theta) + C$$

Now

$$\frac{\partial F}{\partial \theta} = N \iff -r\sin\theta + g'(\theta) = e^{\theta} - r\sin\theta$$

So $g'(\theta) = e^{\theta} \implies g(\theta) = e^{\theta}$

So $\boxed{F(r,\theta) = r\cos\theta + e^{\theta} + C = 0}$

$\iff \boxed{r\cos\theta + e^{\theta} = C}$

$\boxed{2.6}$

(19) $$\frac{dy}{dx} = (x-y+5)^2$$

Let $u = x-y+5$ So $\frac{du}{dx} = 1 - \frac{dy}{dx}$

$$\Leftrightarrow \quad \frac{dy}{dx} = 1 - \frac{du}{dx}.$$

So making these substitutions

$$1 - \frac{du}{dx} = u^2$$

$$\Leftrightarrow \quad \frac{du}{dx} = 1-u^2$$

So separating yields

$$\int \frac{1}{1-u^2}\,du = \int dx$$

So note that $\frac{1}{1-u^2} = \frac{1/2}{1-u} + \frac{1/2}{1+u}$

So $\int \frac{1}{1-u^2}\,du = \frac{1}{2}\int \frac{1}{1-u}\,du + \frac{1}{2}\int \frac{1}{1+u}\,du = -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| + C$

Now we have

$$\frac{1}{2}\ln|1+u| - \frac{1}{2}\ln|1-u| = x + C$$

$$\Longleftrightarrow \quad \ln\left|\frac{\sqrt{1+u}}{\sqrt{1-u}}\right| = x + C$$

$$\Longleftrightarrow \quad \frac{\sqrt{1+u}}{\sqrt{1-u}} = e^{x+c} = e^{x}\cdot e^{c} = ce^{x}$$

Squaring both sides gives

$$\frac{1+u}{1-u} = ce^{2x}$$

$$\Longleftrightarrow \quad 1+u = (1-u)ce^{2x}$$

$$\Longleftrightarrow \quad 1+x-y+5 = (1-x+y-5)\,ce^{2x}$$

$$\Longleftrightarrow \quad x+6+(x+4)\,ce^{2x} = y + yce^{2x}$$

$$\Longleftrightarrow \quad \boxed{y = \frac{x+6+(x+4)ce^{2x}}{1+ce^{2x}}} \quad \text{and} \quad \boxed{y \neq x+4}$$

Remark:

$$1-u \neq 0$$

$$\Leftrightarrow \quad 1-(x-y+5)=0$$

$$\Leftrightarrow \quad y-x-4=0$$

$$\Leftrightarrow \quad \boxed{y=x+4}$$

$\boxed{2.6}$ (21) $\dfrac{dy}{dx}+\dfrac{y}{x}=x^2y^2 \quad (*)$

Bernoulli with $n=2$. So we substitute

$V=y^{1-n}=y^{-1}$. So $\dfrac{dv}{dx}=-y^{-2}\dfrac{dy}{dx}$.

Multiply $(*)$ by $-y^{-2}$ to get

$$-y^{-2}\frac{dy}{dx}-\frac{1}{x}y^{-1}=-x^2$$

Substituting we get

$$\frac{dv}{dx}+\left(-\frac{1}{x}\right)v=-x^2 \qquad (**)$$

$$\mu(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln|x|} = \frac{1}{x}.$$

Multiply (**) by $\mu(x)$ and integrate

$$\mu(x)\, v = \int \mu(x)(-x^2)\,dx$$

$$= -\int x\,dx$$

$$= -\frac{1}{2}x^2 + C$$

So $$v = \frac{-\frac{1}{2}x^2 + c}{\mu(x)} = -\frac{1}{2}x^3 + cx.$$

Plug y^{-1} in for v & solve for y.

Doing so, we get $\boxed{y = \frac{2}{cx - x^3}}$

$\boxed{2.6}$ $\boxed{29}$ $\boxed{\text{Note}}$ Original eqation is exact, but this is to show how to substitute for linear eq.

$$(-3x+y-1)dx + (x+y+3)dy = 0$$

Let $x = u+h$ and $y = v+k$ where h and k are constants satisfying the following system:

$$-3h+k-1=0$$
$$-\,(h+k+3=0)$$

$$-4h-4=0 \Rightarrow h=-1$$

So $k=-2$.

So $x = u-1$ and $y=v-2$. So making this substitution gives the diff. eq.

$$(-3u+v)du + (u+v)dv = 0$$

$$\Leftrightarrow \quad (-3u+v)du = -(u+v)dv$$

$$\Leftrightarrow \quad \frac{du}{dv} = \frac{-(u+v)}{-3u+v} = \frac{u+v}{-v+3u} = \frac{u/v+1}{3u/v-1}$$

Now $N = \frac{\partial F}{\partial v} = u + g'(v) \Leftrightarrow u + v = u + g'(v)$

$\Leftrightarrow g'(v) = v.$

$\Rightarrow g(v) = \frac{1}{2}v^2 + C.$

So

$$-\frac{3}{2}u^2 + uv + \frac{1}{2}v^2 = C.$$

Substituting $u = x+1$ and $v = y+2$ yields

$$-\frac{3}{2}(x+1)^2 + (x+1)(y+2) + \frac{1}{2}(y+2)^2 = C.$$

or equivalently,

$$\boxed{-3(x+1)^2 + 2(x+1)(y+2) + (y+2)^2 = C.}$$

Ch. 2 Review

(35) $(2y^2 + 4x^2)dx - xy\,dy = 0 \qquad y(1) = -2.$

$$\Leftrightarrow \frac{dy}{dx} = \frac{2y^2 + 4x^2}{xy} = \frac{2y}{x} + \frac{4x}{y}$$

Let $v = \frac{y}{x} \Leftrightarrow y = vx \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$

So

$$x\frac{dv}{dx} + v = 2v + \frac{4}{v}$$

$$\Leftrightarrow \quad x\frac{dv}{dx} = v + \frac{4}{v} = \frac{v^2+4}{v}$$

So $$\int \frac{v}{v^2+4}\,dv = \int \frac{1}{x}\,dx$$

$$\Leftrightarrow \quad \frac{1}{2}\ln|v^2+4| = \ln|x| + c$$

(times 2) $$\Leftrightarrow \quad \ln|v^2+4| = \ln|x^2| + c$$

$$\Leftrightarrow \quad v^2+4 = cx^2$$

(subst. for v) $$\Leftrightarrow \quad \frac{y^2}{x^2} = cx^2 - 4$$

$$\Leftrightarrow \quad y^2 = x^2(cx^2-4)$$

So $$y = (+ \text{ or } -)\, x\sqrt{cx^2-4}$$

Since $y(1) = -2 \Rightarrow y = -x\sqrt{cx^2-4}$

So $y(1) = \sqrt{c-4} = -2 \Rightarrow c = 8.$

So $$y = -x\sqrt{8x^2-4} \Leftrightarrow \boxed{y = -2x\sqrt{2x^2-1}}$$

3.2 (3) Want $x(t) :=$ % of acid at time t. in tank

Input (in % of acid): $\frac{6L}{min} \cdot \frac{.2 \text{ parts}}{L} = 1.2$ parts/min

Output: $\frac{8L}{min} \; \frac{x(t) \text{ parts}}{(200-2t) L} = \frac{8}{200-2t} x(t)$ (parts/min)

$$\frac{dx}{dt} = \text{Input} - \text{Output}$$

$$\Leftrightarrow \frac{dx}{dt} = 1.2 - \frac{8}{200-2t} x(t)$$

$$\Leftrightarrow \frac{dx}{dt} + \frac{8}{200-2t} x(t) = 1.2 \qquad (*)$$

Find integrating factor:

$$\mu(t) = e^{\int \frac{8}{200-2t} dt} = e^{4 \int \frac{1}{100-t} dt}$$

$$= e^{-4 \ln|100-t|}$$

$$= \frac{1}{e^{\ln|(100-t)^4|}} = (100-t)^{-4}$$

Multiply both sides of (*) by $\mu(t)$ and integrate to get

$$\mu(t)\,x(t) = 1.2\int (100-t)^{-4}\,dt$$

$$= 1.2\cdot(100-t)^{-3}\left(-\tfrac{1}{3}\right)\cdot(-1) + C$$

$$= .4(100-t)^{-3} + C.$$

Divide both sides by $\mu(t)$ to get

$$x(t) = \left(.4(100-t)^{-3} + c\right)(100-t)^4$$

$$\Longleftrightarrow \quad \boxed{x(t) = .4(100-t) + c(100-t)^4}$$

Finding c:

Since $x(0) = 1$ we have that

$$1 = .4(100) + c(100)^4$$

$$\Longleftrightarrow \quad c = \frac{1-40}{100^4} = -3.9\times10^{-7}.$$

So $\boxed{x(t) = .4(100-t) - (3.9\times10^{-7})(100-t)^4}$

3.2

(15) Let $p(t) :=$ alligator pop. t years after 1980.

So $p_0 = p(0) = 1500$, $p_a = p(13) = 4100$, and
$p_b = p(26) = 6000$, $t_a = 13$, $t_b = 26$. Note: $t_b = 2 \cdot t_a$.

The logistic model of population growth is

$$p(t) = \frac{p_0 \, p_1}{p_0 + (p_1 - p_0)\, e^{-Ap_1 t}} \quad (*)$$

No need to write

~~We know that $p_0 = 1500$ and~~

ignore

$$\cancel{4100 = p(13) = \frac{1500\, p_1}{1500 + (p_1 - 1500)\, e^{-13Ap_1}}}$$

$$\cancel{6000 = p(26) = \frac{1500\, p_1}{1500 + (p_1 - 1500)\, e^{-26Ap_1}}}$$

Since $26 = 2(13)$ $(t_b = 2t_a)$ we can use the results from 3.2 exercise (12) to compute p_1 and A as

$$p_1 = \left[\frac{p_a p_b - 2 p_0 p_b + p_0 p_a}{p_a^2 - p_0 p_b}\right] p_a = 6693.341869$$

$$A = \frac{1}{p_1 t_a} \ln\left[\frac{p_b (p_a - p_0)}{p_0 (p_b - p_a)}\right] = 1.95366582 \times 10^{-5}.$$

Plug these into $(*)$ to get $p(t)$.

3.3

(15)

$$\frac{dT}{dt} = k(M^4 - T^4)$$

$$\Leftrightarrow \quad \int \frac{1}{M^4 - T^4}\, dT = \int k\, dt \qquad (*)$$

Observe that $M^4 - T^4 = (M^2 - T^2)(M^2 + T^2)$

$$= (M-T)(M+T)(M^2+T^2).$$

Using partial fraction decomposition, we find that

$$\frac{1}{M^4 - T^4} = \frac{\frac{1}{4M^3}}{M-T} + \frac{\frac{1}{4M^3}}{M+T} + \frac{\frac{1}{2M^2}}{M^2+T^2}$$

So the left hand side of $(*)$ is equal to

$$-\frac{1}{4M^3}\int \frac{1}{-M+T}\, dT + \frac{1}{4M^3}\int \frac{1}{M+T}\, dT + \frac{1}{2M^2}\int \frac{1}{M^2+T^2}\, dT$$

$$= -\frac{1}{4M^3} \ln|T-M| + \frac{1}{4M^3}\ln|M+T| + \frac{1}{2M^2}\cdot\frac{1}{M}\arctan\left(\frac{T}{M}\right) + C_1$$

Since the right hand side of $(*)$ equals $kt + C_2$

we now have that

So we find that

$$\frac{1}{4M^3} \ln|T-M| = \frac{1}{2M^3} \arctan\left(\frac{T}{M}\right) + \frac{1}{4M^3} \ln|T+M| - kt + C$$

$$\Leftrightarrow \quad \ln|T-M| = 2\arctan\left(\frac{T}{M}\right) + \ln|T+M| - 4M^3kt + C$$

$$\Rightarrow \quad T - M = C(M+T)\, e^{\left(2\arctan\left(\frac{T}{M}\right) - 4M^3kt\right)}.$$