$\boxed{4.2}$

(19) (*) $y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = -3$

Suppose $y = e^{rt}$. Then (*) becomes

$$e^{rt}(r^2 + 2r + 1) = 0$$

$$\leftrightarrow \quad e^{rt}(r+1)^2 = 0$$

$\Rightarrow \quad r = -1$ with multiplicity 2.

So $\quad y = c_1 t e^{-t} + c_2 e^{-t}$

$$y' = -c_1 t e^{-t} + c_1 e^{-t} - c_2 e^{-t}$$

By our original hypothesis,

$$1 = y(0) = c_2$$

$$-3 = y'(0) = c_1 - c_2 = c_1 - 1 \rightarrow c_1 = -2.$$

So $\boxed{y = -2te^{-t} + e^{-t}}$

(21) Given (*) $ay' + by = 0$ observe that for

$y = e^{rt}$ (*) becomes

$$e^{rt}(ar + b) = 0 \Rightarrow r = -\frac{b}{a}.$$

So $\boxed{y = ce^{-\frac{b}{a}t}}$ is a solution to (*).

(23) Now we use our general solution for equations of the form $ay' + by = 0$ with the equation

$$5y' + 4y = 0$$

So $a = 5$, $b = 4$ $\Rightarrow$ $\boxed{y = ce^{-\frac{4}{5}t}}$

$\boxed{4.3}$

(19) $(*)$ $y''' + y'' + 3y' - 5y = 0$

Suppose $y = e^{rt}$. Then $(*)$ becomes

$$e^{rt}(r^3 + r^2 + 3r - 5) = 0$$

$$\Leftrightarrow \quad e^{rt}(r-1)(r^2 + 2r + 5) = 0$$

Using quadratic formula,

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

So $\quad e^{rt}(r-1)(r-(-1+2i))(r-(-1-2i)) = 0$

So $\quad y_1 = c_1 e^{t}$

$y_2 = c_2 e^{-t}\cos 2t + c_3 e^{-t}\sin 2t$

Hence our general solution is given by

$$\boxed{y = c_1 e^{t} + c_2 e^{-t}\cos 2t + c_3 e^{-t}\sin 2t}$$

4.4

11 (*) $y'' + y = 2^x$

Since $2^x = e^{x\ln 2}$, we can rewrite (*) as

(**) $y'' + y = e^{x\ln 2}$.

So we guess that $y = Ae^{x\ln 2}$, for some A. Then (**) becomes

$$Ae^{x\ln 2}\left((\ln 2)^2 + 1\right) = e^{x\ln 2} \rightarrow A = \frac{1}{1+(\ln 2)^2}.$$

So $\boxed{y_p = \frac{1}{1+(\ln 2)^2}e^{x\ln 2}}$ OR $\boxed{y_p = \frac{1}{1+(\ln 2)^2}2^x}$

(15) (*) $y'' - 5y' + 6y = xe^x$

Based on (*) we suppose our particular solution is of the form

$$y_p = (Ax + B)e^x$$

Computing derivatives, we find

$$y_p' = Axe^x + (A+B)e^x$$

$$y_p'' = Axe^x + (2A+B)e^x$$

Substituting these into (*) we get

$$2Axe^x - 3Ae^x + 2Be^x = xe^x$$

which results in the system of equations

$$2A = 1 \Rightarrow A = \tfrac{1}{2}$$

$$-3A + 2B = 0 \Rightarrow B = \tfrac{3}{2}A = \tfrac{3}{2}\left(\tfrac{1}{2}\right) = \tfrac{3}{4}.$$

Hence,

$$\boxed{y_p = \tfrac{1}{2}xe^x + \tfrac{3}{4}e^x}$$

(35) (*) $y''' + y'' - 2y = te^t$

Since $r^3 + r^2 - 2 = (r-1)(r^2+2r+2) = 0$ when $r=1$,

our particular solution has the form

$$y_p = t(Ate^t + Be^t) = At^2e^t + Bte^t.$$

Computing derivatives we find

$$y_p' = At^2e^t + (2A+B)te^t + Be^t$$

$$y_p'' = At^2e^t + (4A+B)te^t + 2(A+B)e^t$$

$$y_p''' = At^2e^t + (6A+B)te^t + (6A+3B)e^t$$

Substituting into (*) we get the equation

$$10Ate^t + (8A+5B)e^t = te^t$$

yielding the system

$$10A = 1 \Rightarrow A = \frac{1}{10}$$

$$8A + 5B = 0 \Rightarrow B = -\frac{8}{5}A = -\frac{4}{25}.$$

Thus, $\boxed{y_p = \frac{1}{10}t^2e^t - \frac{4}{25}te^t}$

4.5

23 (*) $y' - y = 1$; $y(0) = 0$

Suppose $y_h = e^{rt}$. Then substituting into $y' - y = 0$ gives

$$e^{rt}(r-1) = 0 \Rightarrow r = 1$$

So $y_h = ce^t$.

Now based on (*), our particular solution is of the form

$y_p = A$. Substituting into (*) gives

$$-A = 1 \Rightarrow A = -1.$$

So our general solution is

$$y = ce^t - 1$$

Using the initial condition we find

$$0 = y(0) = c - 1 \Rightarrow c = 1.$$

Thus,

$$\boxed{y(t) = e^t - 1}$$

(25) (*) $z'' + z = 2e^{-x}$, $z(0) = 0$, $z'(0) = 0$

Suppose $z_h = e^{rx}$. Substituting into $z'' + z = 0$ gives

$$e^{rx}(r^2+1) = 0 \implies r = \pm i.$$

So $z_h = c_1 \cos x + c_2 \sin x$.

Based on (*) and our auxiliary equation, our particular solution is of the form

$$z_p = Ae^{-x}.$$

Substituting into (*) gives

$$2Ae^{-x} = 2e^{-x} \implies A = 1. \implies z_p = e^{-x}$$

So our general solution has the form

$$z = z_p + z_h = e^{-x} + c_1 \cos x + c_2 \sin x.$$

Using the initial conditions gives

$$0 = z(0) = 1 + c_1 \implies c_1 = -1$$

$$0 = z'(0) = -1 + c_2 \implies c_2 = 1$$

So $\boxed{z(x) = e^{-x} - \cos x + \sin x}$

(39) (*) $y''' + y'' - 2y = te^t + 1$

Suppose $y_h = e^{rt}$. Substituting into $y''' + y'' - 2y = 0$ yields

$$0 = e^{rt}(r^3 + r^2 - 2) = e^{rt}(r-1)(r^2+2r+2)$$
$$= e^{rt}(r-1)(r-(-1+i))(r-(-1-i))$$

So our auxiliary equation has roots $1, -1 \pm i$. To find a particular solution of (*) we now solve

(**) $\quad y''' + y'' - 2y = te^t$

and (***) $\quad y''' + y'' - 2y = 1$

then sum our solutions. Since 1 is a root of auxiliary equation, our particular solution for (**) is of the form

$$y_1 = t(Ate^t + Be^t) = At^2e^t + Bte^t.$$

Observe that we solved (**) in $\boxed{4.4}$ (35) to get

$$y_1 = \frac{1}{10}t^2e^t - \frac{4}{25}te^t$$

To solve (***) note that our particular solution has the form

$$y_2 = A.$$

Substituting into (***) $\Longrightarrow -2A = 1 \Rightarrow A = -\frac{1}{2}.$

Hence, our particular solution is given by $y_p = y_1 + y_2$,

$$\boxed{y_p = -\frac{1}{2} + \frac{1}{10}t^2e^t - \frac{4}{25}te^t}$$

$\boxed{4.6}$

11 (*) $y'' + y = \tan^2 t$

Solving the homogeneous equation gives

$$y_h = c_1 \cos t + c_2 \sin t$$

Letting $y_1 = \cos t$, $y_2 = \sin t$, set

$$y_p = v_1 y_1 + v_2 y_2 .$$

To find v_1 and v_2 we solve the system

$$y_1 v_1' + y_2 v_2' = 0$$
$$y_1' v_1' + y_2' v_2' = \tan^2 t .$$

So we have

$\boxed{1}$ $v_1' \cos t + v_2' \sin t = 0 \Rightarrow v_1' = -v_2' \tan t = \sin t - \tan t \sec t$

$\boxed{2}$ $-v_1' \sin t + v_2' \cos t = \tan^2 t$

$\Leftrightarrow$ $v_2' \sin t \tan t + v_2' \cos t = \tan^2 t$

$\Leftrightarrow$ $v_2' \left(\frac{\sin^2 t}{\cos t} + \cos t \right) = \tan^2 t$

$\Leftrightarrow$ $v_2' \left(\frac{1}{\cos t} \right) = \tan^2 t$

$\Leftrightarrow$ $v_2' = \sin^2 t \cdot \sec t = (1-\cos^2 t) \sec t) = \sec t - \cos t$

So $v_1 = \int (\sin t - \tan t \sec t)\, dt = -\cos t - \sec t + c_1$

$v_2 = \int (\sec t - \cos t)\, dt = \ln|\sec t + \tan t| - \sin t + c_2.$

Setting $c_1 = c_2 = 0$ we get

$$y_p = -(\cos t + \sec t)\cos t + (\ln|\sec t + \tan t| - \sin t)\sin t$$

$$= -\cos^2 t - 1 + \sin t \ln|\sec t + \tan t| - \sin^2 t$$

$$= -(\sin^2 t + \cos^2 t) - 1 + \sin t \ln|\sec t + \tan t|$$

$$= -2 + \sin t \ln|\sec t + \tan t|.$$

Thus, the general solution is given by $y = y_p + y_h$

$$\boxed{y = -2 + \sin t \ln|\sec t + \tan t| + c_1 \cos t + c_2 \sin t}$$

(17) $\frac{1}{2}y'' + 2y = \tan 2t - \frac{1}{2}e^t$

$\iff$ (*) $y'' + 4y = 2\tan 2t - e^t$

Solving the homogeneous equation gives

$$y_h = c_1 \cos 2t + c_2 \sin 2t$$

(the auxiliary equation has roots $\pm 2i$). To determine the particular solution, we solve the equations

(i) $y'' + 4y = -e^t$

(ii) $y'' + 4y = 2\tan 2t$.

Based on the form of (**) and the roots of the auxiliary equation, our guess for a particular solution of (**) is

$$y_i = Ae^t.$$

Substituting into (**) $\Rightarrow$ $5Ae^t = -e^t \to A = -\frac{1}{5}$,

So $y_i = -\frac{1}{5}e^t$.

Now to solve (***) we need to use variation of parameters. This time we will use the formula for v_1 & v_2.

(continued on next page)

Both formulas call for the term

$$y_1 y_2' - y_1' y_2 = (\cos 2t)\cdot(2\cos 2t) - (-2\sin 2t)(\sin 2t)$$
$$= 2\cos^2 2t + 2\sin^2 2t$$
$$= 2.$$

Then, since $a = 1$ (from ⊛)

$$V_1 = \int \frac{-f y_2}{a[y_1 y_2' - y_1' y_2]}\,dt = \int -\frac{2\tan 2t \sin 2t}{2}\,dt$$
$$= -\int (\sec 2t - \cos 2t)\,dt$$
$$= -\tfrac{1}{2}\ln|\sec 2t + \tan 2t| + \tfrac{1}{2}\sin 2t + C_3$$

$$V_2 = \int \frac{f y_1}{a[y_1 y_2' - y_1' y_2]}\,dt = \int \frac{2\tan 2t \cos 2t}{2}\,dt$$
$$= \int \sin 2t\,dt$$
$$= -\tfrac{1}{2}\cos 2t + C_4$$

Setting $C_3 = C_4 = 0$ we simplify the formulations of V_1 & V_2.

So
$$y_{ii} = V_1 y_1 + V_2 y_2$$
$$= \left(-\tfrac{1}{2}\ln|\sec 2t + \tan 2t| + \tfrac{1}{2}\sin 2t\right)\cos 2t + \left(-\tfrac{1}{2}\cos 2t\right)\sin 2t$$
$$= -\tfrac{1}{2}\cos 2t \ln|\sec 2t + \tan 2t|.$$

So our final solution is given by

$$y = y_i + y_{ii} + y_h$$

which is

$$\boxed{y = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{5}e^t - \frac{1}{2}\cos 2t \ln\left| \sec 2t + \tan 2t \right|}$$

4.7

(34) Let $(*)$ $y'' + p(t)y' + q(t)y = g(t)$ be diff eq. w/ solutions t, t^2, t^3.

(a) Define $y_1 = t^2 - t$

$y_2 = t^3 - t$.

Note that for all $r \in \mathbb{R}$, $y_1 \neq r y_2 \Rightarrow$ linearly independent.

Next, since t^3, t^2, and t are solutions to $(*)$ we have that

$$y_1'' + p(t)y_1' + q(t)y_1 = g(t) - g(t) = 0$$

and

$$y_2'' + p(t)y_2' + q(t)y_2 = g(t) - g(t) = 0$$

So y_1 and y_2 are lin. indep. homogeneous solutions to $(*)$.

(b) Since $y_p = t$ is a particular solution to $(*)$, a general solution is given by

$$y = y_p + y_h = t + c_1(t^2 - t) + c_2(t^3 - t).$$

So $y' = 1 + c_1(2t-1) + c_2(3t^2-1)$.

Using the initial conditions we find that

$$\left.\begin{aligned} 2 = y(2) &= 2 + 2c_1 + 6c_2 \\ 5 = y'(2) &= 1 + 3c_1 + 11c_2 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= -6 \\ c_2 &= 2 \end{aligned}$$

So $\boxed{y = t - 6(t^2 - t) + 2(t^3 - t)}$ OR $\boxed{y = 2t^3 - 6t^2 + 5t}$

(45) $t^2y'' - 2ty' - 4y = 0$, $t>0$, $f(t)=t^{-1}$.

Since $t>0$, we can divide through by t^2 to get

(*) $y'' - \frac{2}{t}y' - \frac{4}{t^2}y = 0$.

Recall the formula for reduction of order:

$$y_2(t) = y_1(t)\int \frac{e^{-\int p(t)dt}}{y_1(t)^2}\,dt$$

where $y_1(t)$ is a solution to (*), in our case $y_1(t) = t^{-1}$, and $p(t) = -\frac{2}{t}$ from (*).

First, we find

$$e^{-\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2.$$

So

$$y_2 = t^{-1}\int \frac{t^2}{t^{-2}}\,dt = t^{-1}\int t^4\,dt$$
$$= t^{-1}\left(\frac{1}{5}t^5 + C\right) = \frac{1}{5}t^4 + \frac{C}{t}.$$

By letting $C=0$, we get $y_2 = \frac{1}{5}t^4$ is another linearly independent solution of (*).

(If you want the answer in the back you can multiply by 5 to get t^4 is a solution to (*) that is lin. indep. of t^{-1}.)

(47) $tx'' - (t+1)x' + x = 0, \quad t > 0 \qquad f(t) = e^t.$

Since $t > 0$, we can divide through by t to get

$$x'' - (1 + t^{-1})x' + \frac{1}{t}x = 0.$$

Using reduction of order with $p = -(1+t^{-1})$ and $y_1(t) = e^t$ we get:

$$e^{-\int p(t)dt} = e^{\int (1+t^{-1})dt} = e^{t+\ln t} = te^t$$

and

$$y_2 = y_1 \int \frac{e^{-\int p(t)dt}}{y_1^2} dt$$

$$= e^t \left(\int \frac{te^t}{e^{2t}} dt \right)$$

$$= e^t \left(\int te^{-t} dt \right)$$

$$= e^t \left(-te^{-t} + \int e^{-t} dt \right) = e^t \left(-te^{-t} - e^{-t} + c \right)$$

So $y_2 = -t - 1 + ce^t$. Letting $c = 0$ we get

$\boxed{y_2 = -t - 1!}$ (If you want answer in back of book multiply by -1)