VIEWING MODULAR FORMS AS AUTOMORPHIC REPRESENTATIONS

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These notes answer the question "How does the classical theory of modular forms connect with the theory of automorphic forms on GL_2?" They are a more detailed version of talks given at a student reading group based on Jacquet and Langland's book [JL70]. They were given after the first ten sections had been covered, which had discussed the local representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$, $\mathrm{GL}_2(\mathbb{R})$, and $\mathrm{GL}_2(\mathbb{C})$, and then defined the global Hecke algebra and automorphic forms and representations. The aim was to connect the theory which had been developed with something more familiar, and understand the motivation behind the theory which had been developed.

Given a cuspidal holomorphic modular form $f(z)$ of weight k, level N, and nebentypus character χ, we will describe how to associate an automorphic form $F(g)$ on $\mathrm{GL}_2(\mathbb{R})^+$ and an automorphic form $\varphi_f(g)$ on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. We attempt to collect all of the details necessary to understand what properties $F(g)$ and $\varphi_f(g)$ possess, and make clear how these properties of automorphic forms relate to classical properties of modular forms. In the case when $f(z)$ is an Hecke eigenform, we will also show there is an automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated to f containing φ_f, and describe as much as possible its local components. In the process of doing so, we will explain how the data of the Hecke eigenvalues is encoded in the local representations. This will explain the sense in which the Hecke eigenvalues are local information, a fact that is not obvious in the classical setting. It is possible obtain information about most of the local components by pure thought. At primes that divide the level N, the situation is more complicated as supercuspidal representations can arise. Loeffler and Weinstein describe the situation and an algorithm to deal with these cases [LW12].

Example 0.1. For example, one of the newforms of level 99 with trivial nebentypus and weight 2 has q-expansion

$$f(z) = q - q^2 - q^4 - 4q^5 - 2q^7 + 3q^8 + 4q^{10} - q^{11} + \dots$$

Using the algorithm as implemented in SAGE, we find that

- The infinite component is a discrete series of weight 2.
- At $p = 2$, the local component is an unramified principal series. The two unramified characters send uniformizers to the roots of $x^2 + x + 2$. Note that the Hecke eigenvalue for T_2 is -1.
- At $p = 3$, the local component is a supercuspidal representation of conductor 2. Sage computes a description in terms of two characters of extensions of $\mathbb{Q}_p$.
- At $p = 11$, the local component is a special representation of conductor one. It is the twist of the Steinberg by the unramified character of $\mathbb{Q}_{11}^\times$ of order two. Note that the Hecke eigenvalue is -1.

These notes assume the reader is familiar with the representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$, $\mathrm{GL}_2(\mathbb{R})$, and $\mathrm{GL}_2(\mathbb{C})$ over the complex numbers, in particular the classification of irreducible smooth admissible representations. We also assume knowledge of the definition of automorphic forms and representations on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, local and global Hecke algebras and the manner in which automorphic representations decompose as restricted tensor products. This is essentially the material in the first ten sections of Jacquet and Langland [JL70]. Other standard references for some or all of this material include books by Bump [Bum97], Bushnell and Henniart [BH06], and Gelbart [Gel75].

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In Section 1, we relate quotients of the upper half plane to double coset spaces for $\mathrm{GL}_2(\mathbb{R})^+$ and $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ as a preliminary step in transporting modular forms to $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Section 2 then constructs the automorphic forms $F(g)$ and $\varphi_f(g)$ associated to a classical modular form f and establishes their important properties. Using these automorphic forms, one can construct the automorphic representation associated to f. In order to do so, we first review some facts about the local representation theory of $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ in Sections 3 and 4. Besides a general review of the classification results, we also include material that was not touched upon in the seminar (or in [JL70]). In particular, Section 3 discusses $(\mathfrak{g}, K)$ modules and the local Hecke algebra from a different perspective than [JL70], and connects the action of $U(\mathfrak{gl}_2(\mathbb{C}))$ with explicit differential operators on $C^\infty(\mathrm{GL}_2(\mathbb{R}))$. Section 4 includes information about spherical representations and the spherical Hecke algebra. Finally in Section 5 we use the multiplicity one theorem to construct an irreducible automorphic representation π_f containing φ_f and describe the local components as much as is possible.

1. Preliminaries on the Upper Half Plane and Quotients

In this section, we will relate quotients of the upper half plane $\mathcal{H}$ with quotients of $\mathrm{GL}_2(\mathbb{R})$ and with quotients of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. We consider the group GL_2 with center Z. The points over $\mathbb{R}$ with positive determinant, $\mathrm{GL}_2(\mathbb{R})^+ = \{g \in \mathrm{GL}_2(\mathbb{R}) : \det(g) > 0\}$, act on the upper half plane $\mathcal{H}$ via fractional linear transformations.

The connection between functions on the upper half plane and functions on $\mathrm{GL}_2(\mathbb{R})$ boils down to the following observation: $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}$ with stabilizer $\mathrm{SO}_2(\mathbb{R})$. It is technically better to work with $\mathrm{GL}_2(\mathbb{R})^+$ and consider separately $K = \mathrm{SO}_2(\mathbb{R})$ and the center $Z(\mathbb{R})^+ \subset \mathrm{GL}_2(\mathbb{R})^+$.

This idea and the Iwasawa decomposition give a convenient system of coordinates to use on $\mathrm{GL}_2(\mathbb{R})^+$. Because $\mathrm{GL}_2(\mathbb{R})^+$ is the product of the maximal compact subgroup K and the Borel of upper triangular matrices, every element can be expressed in the form

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} y^{1/2} & y^{-1/2}x \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{1.1}$$

Letting this act on the upper half plane, we see that i is sent to $x + iy$. Therefore using (x, y, θ, λ) as coordinates on $\mathrm{GL}_2(\mathbb{R})^+$ (subject to the obvious restrictions that $\lambda > 0$, $y > 0$, and $\theta \in [0, 2\pi)$) gives a convenient coordinate system that is related to the standard coordinates on the upper half plane in the sense that $g \cdot i = x + iy$.

Remark 1.1. This is a different coordinate system than is used in Bump's book. In particular, Bump uses the matrix $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ for his rotation matrix. This is an unnatural direction to rotate, but simplifies certain formulas. In particular, actions of K by $e^{-ik\theta}$ become actions by $e^{ik\theta}$ such as in (3.1) and the action of K on automorphic forms, and negative signs are removed in the differential operators for Δ and L in Section 3.1.

We now turn to $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Recall that the strong approximation theorem states that for any compact open subgroup $K \subset \mathrm{GL}_2(\mathbb{A}_\mathrm{f})$ such that $\det(K) = \mathbb{A}_\mathrm{f}^\times$, we have $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})\,\mathrm{GL}_2(\mathbb{R})K$. (This relies on the class number of $\mathbb{Q}$ being one.) A convenient compact open subgroup to work with will be

$$K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : c \equiv 0 \mod N \right\}.$$

The connection between the $\mathrm{GL}_2(\mathbb{R})^+$ (or $\mathrm{SL}_2(\mathbb{R})$) and $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ is the following proposition.

Proposition 1.2. *For any positive integer N, there are natural isomorphisms*

$$\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{R}) \simeq Z(\mathbb{A}_\mathbb{Q})\,\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N)$$
$$\Gamma_0(N)\backslash \mathrm{GL}_2(\mathbb{R})^+ \simeq \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N).$$

Remark 1.3. Adding in an archimedean component to $K_0(N)$ such as $\mathrm{SO}_2(\mathbb{R})$ would give a more direct comparison with the upper half plane.

Proof. As $\mathrm{GL}_2(\mathbb{Q})$ contains elements with negative determinant, we may modify the strong approximation statement to conclude that

$$\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})\,\mathrm{GL}_2(\mathbb{R})^+ K_0(N).$$

Now consider the map given by including $\mathrm{GL}_2(\mathbb{R})^+$ at the archimedean place and passing to the quotient:

$$\mathrm{GL}_2(\mathbb{R})^+ \to \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N).$$

It is surjective by the strong approximation theorem.

Now let us analyze how close to being injective it is. Let g_∞ and g'_∞ have the same image. This means there exist $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ and $k_0 \in K_0(N)$ such that $g'_\infty = \gamma g_\infty k_0$. The element γ is embedded diagonally, so separate the real and finite adelic parts, writing $\gamma = \gamma_\infty \gamma_f$ with $\gamma_\infty \in \mathrm{GL}_2(\mathbb{R})$ and $\gamma_f \in \mathrm{GL}_2(\mathbb{A}_\mathrm{f})$. As g_∞ and g'_∞ are completely in the archimedean place, we see that

$$g'_\infty = \gamma_\infty g_\infty \quad \text{and} \quad \gamma_f = k_0^{-1}.$$

The first says $\det(\gamma_\infty) > 0$, and with the second it implies $\gamma_f \in K_0(N) \cap \mathrm{GL}_2(\mathbb{Q}) = \Gamma_0(N)$. Therefore g'_∞ and g_∞ differ by an element of $\Gamma_0(N)$. This proves the second isomorphism.

For the first, we just add the centers. Strong approximation (for $\mathbb{G}_m$ in this case) says that

$$Z(\mathbb{A}_\mathbb{Q}) = Z(\mathbb{R})^+ Z(\mathbb{Q})\left(Z(\mathbb{A}_\mathbb{Q}) \cap K_0(N)\right).$$

Therefore we obtain an isomorphism

$$Z(\mathbb{R})^+\Gamma_0(N)\backslash\,\mathrm{GL}_2(\mathbb{R})^+ \simeq Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N).$$

But the left is isomorphic to $\Gamma_0(N)\backslash\,\mathrm{SL}_2(\mathbb{R})$. □

2. Constructing the Automorphic Form Associated to a Modular Form

In this section, we will construct automorphic forms on $\mathrm{GL}_2(\mathbb{R})^+$ and $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated to a holomorphic modular form. We are mainly interested in the adelic viewpoint, but constructing the automorphic form adelically boils down to constructing it at the archimedean place and then making a slight modification at the non-archimedean ones. The basic idea is to take the identification of spaces in the previous section and set up a correspondence between functions on the upper half plane, $\mathrm{GL}_2(\mathbb{R})^+$, and $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Instead of actually working with the quotients, we will work with functions which satisfy transformation laws.

In the course of the construction, we check the construction satisfies all of the requirements to be an automorphic form to illustrate how the classical properties of modular forms appear in the automorphic language as conditions such as K-finiteness.

Remark 2.1. The same story is true for Maass forms. For simplicity, we focus just on the case of holomorphic modular forms.

2.1. Classical Modular Forms.

Let $f(z)$ be a modular form of level N and weight $k \geq 2$ with nebentypus character χ. Recall that this means:

(i) f is a holomorphic function on the upper half plane $\mathcal{H} = \{z : \mathrm{Im}(z) > 0\}$.

(ii) For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z). \tag{2.1}$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$, recall one defines $j(\gamma, z) = (cz+d)$ and the slash operator by $(g|_k\gamma)(z) = \det(\gamma)^{k/2} j(\gamma,z)^{-k} g(\gamma \cdot z)$. So this transformation rule can be rewritten as $f|_k\gamma = \chi(d) f$ for $\gamma \in \Gamma_0(N)$.

(iii) f is holomorphic at the cusps. If the cusp is infinity, this means that the q-expansion (where $q = e^{2\pi i z}$) is of the form

$$f(z) = \sum_{n \geq 0} a_n q^n.$$

For a general cusp, pick $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ which takes infinity to that cusp ask that $f|_k\gamma$ be holomorphic at infinity. However, it is more convenient just to require that $f|_k\gamma$ be holomorphic at infinity for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Recall that the coefficients a_n are Fourier coefficients, so may be computed (for any $z \in \mathcal{H}$) via the integrals

$$a_n = \int_0^1 f(z+t) e^{-2\pi i n t} dt. \tag{2.2}$$

The theory of modular forms is discussed in many places. Chapter 1 of Bump [Bum97] presents the subject with a view towards connecting it with automorphic forms.

2.2. **Automorphic Forms for** $\mathrm{GL}_2(\mathbb{R})$**.** The "classical" theory of automorphic forms for $\mathrm{GL}_2(\mathbb{R})$ is discussed in Chapter 2 of Bump [Bum97]. It is important motivation and technical ingredient in the adelic theory (for example, spectral theory gives a decomposition of the space of automorphic forms which is then translated to the adelic setting), but we will not assume knowledge of it. We will just discuss how a modular form is an example of an automorphic form on $\mathrm{GL}_2(\mathbb{R})$ before moving on the adelic situation.

The observation that $\mathrm{GL}_2(\mathbb{R})^+$ acts on the upper half plane with stabilizer $K = \mathrm{SO}_2(\mathbb{R})$ suggests how to connect modular forms on the upper half plane and automorphic forms on $\mathrm{GL}_2(\mathbb{R})^+$. Given a cusp form f, we consider the function defined on $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$ by

$$F(g) := (f|_k g)(i) = (ad-bc)^{k/2} (ci+d)^{-k} f\left(\frac{ai+b}{ci+d}\right).$$

This is the automorphic form for $\mathrm{GL}_2(\mathbb{R})$ associated to f. It has many nice properties.

(i) For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ it satisfies

$$F(\gamma g) = (f|_k \gamma g)(i) = \chi(d)(f|_k g)(i) = \chi(d) F(g).$$

(ii) For $\kappa = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in K = \mathrm{SO}_2(\mathbb{R})$, we see that

$$\det(g\kappa)^{k/2} j(g\kappa, i)^{-k} = \det(g)^{k/2} j(g,i)^{-k} e^{-ik\theta}$$

as K is the stabilizer of i. Hence we see that

$$F(g\kappa) = \det(g)^{k/2} j(g\kappa, i)^{-k} f\left(g \cdot (\kappa \cdot i)\right) = e^{-ik\theta} F(g).$$

Note that this implies F is K-finite.

(iii) For $\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \in Z$, we see

$$F(\gamma g) = (f|_k \gamma g)(i) = (\lambda^2)^{k/2} \det(g)^{k/2} \lambda^{-k} j(g,i)^{-k} f(g \cdot i) = \omega(\gamma) F(g)$$

where $\omega(\gamma)$ is 1 when $\lambda > 0$ and is $\chi(-1)$ when $\lambda < 0$. Note that $\chi(-1)$ is arising because of subtleties arising from the choice of square root involved in defining $\det(g)^{k/2}$: these are already present in the classical theory of modular forms and are dealt with the same way.

(iv) We show the function F is bounded (if f was a modular form but not a cusp form, it would be of moderate growth). First, a calculation shows that

$$\operatorname{Im}\left(\frac{ai+b}{ci+d}\right)^{k/2} = \left(\frac{\det(g)}{|ci+d|^2}\right)^{k/2}$$

so it suffices to show that $|\operatorname{Im}(z)|^{k/2}|f(z)|$ is bounded. At infinity, using the q-expansion we see that for $\operatorname{Im}(z) > c$

$$|f(z)| = |a_1 q + a_2 q^2 + \dots| \leq C'|q|$$

where the constant C' depends on c. But $|q| = e^{-2\pi \operatorname{Im}(z)}$, so $|\operatorname{Im}(z)|^{k/2}|f(z)|$ is bounded and goes to zero as $\operatorname{Im}(z)$ goes to infinity. Using the action of $\mathrm{SL}_2(\mathbb{Z})$, one can do a similar analysis for any cusp. Then we use knowledge about fundamental domains in the upper half plane (or the corresponding facts about Siegel domains for GL_2) to reduce to this case by covering by neighborhoods that is of this form around each cusp.

(v) Recall that the Lie algebra $\mathfrak{gl}_2(\mathbb{R})$ acts on the space of smooth functions on $\mathrm{GL}_2(\mathbb{R})$ via the derivative of the right translation action. So does its universal enveloping algebra $U(\mathfrak{gl}_2(\mathbb{C}))$. This action commutes with a Laplacian operator Δ which can be defined either by writing down an element in the center of the universal enveloping algebra or abstractly (the negative of the Killing form gives a Riemannian metric on $\mathrm{GL}_2(\mathbb{R})$, and there is an associated Laplacian). We defer the extensive calculations to show that F is an eigenfunction of Δ with eigenvalue $\frac{k}{2}\left(1-\frac{k}{2}\right)$ to the analysis leading up to Corollary 3.3.

(vi) We finally interpret the condition of vanishing at the cusps. If a modular form $f(z)$ vanishes at the cusp at infinity, by (2.2) it follows that for all $z \in \mathcal{H}$

$$\int_0^1 f(z+t)dt = 0.$$

In light of (1.1), elements of the form

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} y^{1/2} & y^{-1/2}x \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

send i to $x+iy$. We calculate using (ii) and (iii) that

$$F(g) = (f|_k g)(i) = \omega(\gamma) e^{-ik\theta} y^{k/2} f(z). \tag{2.3}$$

Furthermore, unwinding the left action of the unipotent matrix $n_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ we see

$$F(n_t g) = \omega(\gamma) e^{-ik\theta} y^{k/2} f(z+t).$$

Therefore the cuspidality condition is equivalent to

$$\int_0^1 F(n_t g)dt = 0$$

for all $g \in \mathrm{GL}_2(\mathbb{R})^+$. Using the action of $\mathrm{SL}_2(\mathbb{Z})$ on the cusps, there is a similar statement for other cusps.

This construction is discussed in [Bum97, Section 3.2]: in terms of the theory of automorphic forms on $\mathrm{GL}_2(\mathbb{R})$, this shows that F lies in the space of cusp forms $\mathcal{A}_0(\Gamma_0(N)\backslash \mathrm{GL}_2(\mathbb{R})^+, \chi, \omega)$.

2.3. **Automorphic Forms for** $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$**.** We now describe how to associate to f and F an automorphic form φ_f on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. The idea is similar to that of $\mathrm{GL}_2(\mathbb{R})^+$: we use Proposition 1.2 to see a relation between the spaces and then connect functions on $\mathrm{GL}_2(\mathbb{R})$ with functions on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ that satisfy certain transformation properties. The adelic perspective makes it much clearer that modular forms are arithmetic in nature and have local information at each prime.

Given a cusp form f, we define a function φ_f on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ as follows. We use strong approximation to write an element of $\mathrm{GL}_2(\mathbb{A})$ as a product where $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $g_\infty \in \mathrm{GL}_2(\mathbb{R})^+$, and $k_0 \in K_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : c \equiv 0 \mod N\}$. Then we define

$$\varphi_f(\gamma g_\infty k_0) := F(g_\infty)\lambda(k_0) = (f|_k g_\infty)(i)\lambda(k_0).$$

The function λ is an adelization of the Dirichlet character χ. Since $f(z)$ is not quite invariant under $\Gamma_0(N)$, in light of Proposition 1.2 it is no surprise that φ_f needs to incorporate a correction from χ in order to end up $K_0(N)$-invariant. We will first define λ and then check φ_f is well defined.

The character λ is defined to in two steps: first associate an adelic character ω to χ (the grossencharacter), then convert this to a character of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$.

Defining ω is an exercise in class field theory. The strong approximation theorem states that

$$\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \cdot \mathbb{R}_{>0}^\times \cdot \prod_p \mathbb{Z}_p^\times$$

Using the Chinese remainder theorem, we realize $\mathbb{Z}/N\mathbb{Z}^\times$ as a quotient of $\prod_p \mathbb{Z}_p^\times$. Thus composing with the *inverse* of χ gives us the grossencharacter $\omega = \prod_p \omega_p : \mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ that is trivial on $\mathbb{R}_{>0}^\times$. More details are found in [Bum97, Proposition 3.1.2]. (Another way to express this is to realize $\mathbb{Z}/N\mathbb{Z}^\times$ as the ray class group over $\mathbb{Q}$ with modulus $\mathfrak{m} = N\infty$.)

To get a character of $K_0(N)$, we define

$$\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \omega(d) = \prod_{p|N} \omega_p(d_p)$$

where d_p denotes the $\mathbb{Q}_p^\times$ component of d. Let $\pi_p \in \mathbb{A}_\mathbb{Q}^\times$ be the image of p under the inclusion $\mathbb{Q}_p^\times \hookrightarrow \mathbb{A}_\mathbb{Q}^\times$. We record some properties of ω and λ.

Lemma 2.2. *If $p \nmid N$, we have $\omega|_{\mathbb{Z}_p^\times} = 1$ and $\omega(\pi_p) = \chi(p)$. The archimedean component ω_∞ is trivial on $\mathbb{R}_{>0}^\times$. Thus if d is a positive integer prime to N, $\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \chi(d)^{-1}$.*

Proof. By construction $\omega_{\mathbb{Z}_p^\times} = 1$. Using the decomposition from the strong approximation theorem, we can write

$$\pi_p = p \cdot 1 \cdot \alpha$$

where $\alpha \in \prod_p \mathbb{Z}_p^\times$ with $\alpha_v = 1/p$ for $v \neq p$ and $\alpha_p = 1$. Note that α reduces to $1/p$ in $(\mathbb{Z}/N\mathbb{Z})^\times$. Then $\omega(\pi_p) = \left(\chi(\frac{1}{p})\right)^{-1} = \chi(p)$.

By construction ω_∞ is trivial on $\mathbb{R}_{>0}^\times$. To prove the last claim, it suffices to show that

$$\chi(d) = \prod_{p|N} \omega_p^{-1}(d_p).$$

But as ω is trivial on $\mathbb{Q}^\times$, the right side is equal to $\prod_{p\nmid N} \omega_p(d_p)$. If d were a rational prime relatively prime to N, the first part of the lemma would imply this equals $\chi(d)$. Then extend by multiplicativity. □

We now return to analyzing φ_f. The properties we established for the automorphic form F on $\mathrm{GL}_2(\mathbb{R})$ associated to f contain most of the work.

Proposition 2.3. *The function φ_f is well defined. It is an automorphic form with central character ω, and is a cusp form.*

Proof. To check it is well defined, consider two decompositions $\gamma g_\infty k_0 = g = \gamma' g'_\infty k'_0$. Looking at the archimedean component of g gives $\gamma_\infty g_\infty = \gamma'_\infty g'_\infty$, so $\det(\gamma_\infty^{-1}\gamma'_\infty) > 0$. The finite components give that

$$\gamma_f k_0 = \gamma'_f k'_0,$$

so $\gamma_f^{-1}\gamma'_f \in K_0(N)$. Hence $\gamma^{-1}\gamma' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} =\in K_0(N) \cap \mathrm{GL}_2(\mathbb{R})^+ = \Gamma_0(N)$. Then the transformation law for F implies that

$$F(g_\infty) = F(\gamma^{-1}\gamma' g'_\infty) = \chi(d) F(g'_\infty).$$

But by the lemma, $\chi(d)^{-1} = \lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \lambda(\gamma^{-1}\gamma') = \lambda(k_0(k'_0)^{-1})$. Hence

$$F(g'_\infty)\lambda(k'_0) = F(g_\infty)\lambda(k_0).$$

Thus φ_f is well defined.

To check that it is an automorphic form, there are a number of things to verify. It is smooth, because F is smooth in the archimedean sense and λ is locally constant. The following list of properties is based on the properties of F given in Section 2.2.

(i) Left invariance under $\mathrm{GL}_2(\mathbb{Q})$ follows by definition. (But note well-definedness used the left invariance of F.)

(ii) Taking $K = K_0(N)\,\mathrm{SO}_2(\mathbb{R})$, for $k = k_0 k_\infty \in K$ we have that

$$\varphi_f(gk) = F(g_\infty k_\infty)\lambda(kk_0) = e^{-ik\theta} F(g_\infty)\lambda(k)\lambda(k_0) = e^{-ik\theta}\lambda(k_0)\varphi_f(g) \tag{2.4}$$

In particular, it is K-finite.

(iii) For $g \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ and $z \in \mathbb{A}_\mathbb{Q}^\times$, we can check that

$$\varphi_f\left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} g\right) = \omega(z)\varphi_f(g).$$

This is immediate for $z \in \mathbb{Q}^\times$, $z \in \mathbb{R}^\times_{>0}$, and $z \in \mathbb{Z}_p^\times$, and follows in general using strong approximation.

(iv) φ_f is bounded because we know that F is.

(v) Letting $\mathcal{Z}$ be the center of the universal enveloping algebra of $\mathfrak{gl}_2(\mathbb{R})$, φ_f is $\mathcal{Z}$-finite. This is because the action of $\mathcal{Z}$ is just in the archimedean component and we know that F is $\mathcal{Z}$-finite (and furthermore is an eigenfunction).

(vi) The cuspidality condition is that for all $g \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$,

$$\int_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}^\times} \varphi_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0.$$

To evaluate the integral on the left, we use $\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times \simeq \mathbb{R}^\times_{>0}\cdot\widehat{\mathbb{Z}}^\times$ and write the quotient measure as a product. But at the archimedean place the integral is 0 because we know that F is cuspidal.

□

Remark 2.4. In fact, this construction gives an isomorphism between $S_k(N,\chi)$ and the space of functions on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ satisfying certain properties: see [Gel75, Proposition 3.1] for a precise statement.

Thus from a cuspidal holomorphic modular form f, we have constructed a cuspidal automorphic form $\varphi_f \in \mathcal{A}_0(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}), \omega)$.

The next thing to do is to analyze the automorphic representation generated by φ_f. We must first review the local archimedean and non-archimedean representation theory.

3. Background at the Archimedean Places

In this section we will review material about representations of $\mathrm{GL}_2(\mathbb{R})$, starting with how the Lie algebra acts and what this action looks like in the coordinates of (1.1). We then review the classification of $(\mathfrak{g}, K)$-modules and the local Hecke algebra.

3.1. **The Action of $\mathfrak{gl}_2(\mathbb{R})$.** Recall that via the adjoint representation, the Lie algebra $\mathfrak{gl}_2(\mathbb{R})$ acts on $\mathrm{GL}_2(\mathbb{R})$ and hence gives an action of the space of smooth functions on $\mathrm{GL}_2(\mathbb{R})$ via right translation. For $X \in \mathfrak{gl}_2(\mathbb{R})$ and a smooth function f, the action is given by

$$(X \cdot f)(g) = \frac{d}{dt} f(g \exp(tX))|_{t=0}.$$

In the case of GL_2, as $I + tX$ is a path in $\mathrm{GL}_2(\mathbb{R})$, for small t, tangent to the one parameter subgroup $\exp(tX)$, an alternate expression is

$$(X \cdot f)(g) = \frac{d}{dt} f(g(1 + tX))|_{t=0}.$$

The following elements in $\mathfrak{gl}_2(\mathbb{R})$ are important:

$$\hat{R} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Using these, we can define an element in the universal enveloping algebra

$$\Delta = \frac{-1}{4} \left(\hat{H}^2 + 2\hat{R}\hat{L} + 2\hat{L}\hat{R} \right).$$

One shows that Δ is in the center. In fact, the center is a polynomial algebra generated by Δ and I. The action of Δ gives the Laplace operator. We will sketch how one derives an explicit description of this operator in terms of the (x, y, θ, λ) coordinate system on $\mathrm{GL}_2(\mathbb{R})^+$.

To do calculations, it is also convenient to work with related elements in the complexification:

$$R = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad L = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad H = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Delta = -\frac{1}{4}(H^2 + 2RL + 2LR)$$

These are related via the Cayley transform to $\hat{R}, \hat{L} \ldots$. Recall the Cayley transform is conjugation by

$$\mathcal{C} = -\frac{1+i}{2} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}.$$

It corresponds to the transformation taking the upper half plane into the unit disc. The actions of R and L are easier to compute. We will sketch a derivation of a formula for Δ using this.

Proposition 3.1. *On the space of smooth functions, the action of Δ is given by*

$$-y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial^2}{\partial x \partial \theta}.$$

Proof. The idea is simple: write down the exponential for an element like $\hat{R}$ or H and evaluate the derivative. Doing calculations is unavoidable, but necessary for what we wish to do.

For example, let us do H. First consider $W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. By definition, for a function f and $g \in \mathrm{GL}_2(\mathbb{R})^+$, $(W \cdot f)(g) = \frac{d}{dt} f(gk_t)|_{t=0}$. By evaluating the matrix powers in the definition of the exponential, we see that

$$\exp(tW) = \kappa_t = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

Expressing g in (x, y, θ, λ) coordinates using (1.1), we have

$$g \exp(tW) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} y^{1/2} & y^{-1/2}x \\ 0 & y^{-1/2} \end{pmatrix} \kappa_{\theta+t}$$

This is $(x, y, \theta+t, \lambda)$ in that coordinate system, so the derivative evaluated at zero is $\frac{\partial}{\partial\theta} f(g)$. Using the action of W, one immediate gets the action of H.

Likewise, we can easily deal with $\hat{R}$ in the case of g having $\theta = 0$. For then

$$g \exp(t\hat{R}) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} y^{1/2} & y^{-1/2}x \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} y^{1/2} & y^{-1/2}(x+yt) \\ 0 & y^{-1/2} \end{pmatrix}.$$

This has coordinates $(x + yt, y, 0, \lambda)$, so $\hat{R}$ sends f to

$$y \frac{\partial f}{\partial x}(g)$$

If θ is not zero, things are more complicated because one needs to rearrange the product. The Cayley transform helps with this. Additional calculations can be found in [Bum97, Proposition 2.2.5]. Note that Bump uses $-\theta$ for a coordinate in the rotation matrices which changes some signs. These calculations give the formula for Δ. $\square$

Remark 3.2. For future use, we also record that the action of L is given by

$$e^{2i\theta} \left(-iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{1}{2i} \frac{\partial}{\partial \theta} \right)$$

If we had worked in more generality and identified Maass forms of weight k with automorphic forms, the operators R and L would correspond to the raising and lowering operators for Maass forms.

Using this description, we can finally show that the automorphic form F on $\mathrm{GL}_2(\mathbb{R})$ we associated to a cusp form is an eigenfunction of Δ.

Corollary 3.3. *Let f be a holomorphic modular form of weight k and F the associated automorphic form on $\mathrm{GL}_2(\mathbb{R})^+$. Then*

$$\Delta F = \frac{k}{2} \left(1 - \frac{k}{2} \right) F$$

Proof. Unraveling the definition of F, we compute that

$$F(g) = (f|_k g)(i) = e^{-ik\theta} y^{k/2} f(x + iy)$$

using the coordinates in (1.1). Now we compute that

$$\Delta F = -e^{-ik\theta} \left(y^{k/2+2} f_{xx} + y^{k/2+2} f_{yy} + k y^{k/2+1} f_y + \frac{k}{2} \left(\frac{k}{2} - 1 \right) y^{k/2} f + y(-ik) y^{k/2} f_x \right).$$

Since f is holomorphic, $f_{xx} + f_{yy} = 0$ and $if_x = f_y$. Therefore we have that

$$\Delta F = -e^{-ik\theta} \frac{k}{2} \left(\frac{k}{2} - 1 \right) y^{k/2} f = \frac{k}{2} \left(1 - \frac{k}{2} \right) F.$$

$\square$

3.2. $(\mathfrak{g}, K)$**-Modules.** Let $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$ and $K = O_2(\mathbb{R})$. Recall that a $(\mathfrak{g}, K)$-module for $\mathrm{GL}_2(\mathbb{R})$ is a vector space V with representations π_K and $\pi_\mathfrak{g}$ of K and $\mathfrak{g}$ such that

(1) The representations of $\mathfrak{g}$ and K are compatible in the sense that an element $X \in \mathrm{Lie}(K)$ acts the same way using $\pi_\mathfrak{g}$ action and through the infinitesimal action of K on V (this refers to the action $d\pi_K$ of $\mathrm{Lie}(K)$ on V).
(2) V is an algebraic direct sum of finite dimensional irreducible representations of K
(3) For $X \in \mathfrak{g}$ and $k \in K$ we have $\pi_K(k)\pi_\mathfrak{g}(X)\pi_K(k^{-1}) = \pi_\mathfrak{g}(\mathrm{Ad}(k)X)$ as operators on V.

Such a module is admissible if each irreducible representation of K appears with finite multiplicity.

Let σ_k be the representation of $\mathrm{SO}_2(\mathbb{R})$ given by sending a matrix with angle θ to $e^{-ik\theta}$. Then any $(\mathfrak{g}, K)$ module V decomposes as a direct sum

$$V = \bigoplus_k V(k) \tag{3.1}$$

where $V(k)$ is the isotypic component of V corresponding to σ_k. Calculations with the Lie algebra show that $\hat{R}$ and $\hat{L}$ raise and lower k by two. Similarly, H acts by k on $V(k)$ and Λ acts by $\frac{k}{2}(1 - \frac{k}{2})$. For details, see [Bum97, Proposition 2.5.2]. There is also a classification of irreducible $(\mathfrak{g}, K)$ modules in [Bum97, Theorem 2.5.5] or [JL70, Section 5].

- If the representation is finite dimensional, it is a twist of the natural representation on degree n homogeneous polynomials in two variables. In this case $V(n-2), V(n-4), \ldots, V(2-n)$ are one dimensional, and all the other $V(k)$ are zero. (This is the symmetric power of the standard representation.) Twisting means multiplying by $\chi \circ \det$ where χ is a character of $\mathbb{R}^\times$.
- We could have
$$V = \bigoplus_{k \equiv \epsilon \mod 2} V(k)$$
where the $V(k)$ appearing are one dimensional and $\epsilon = 0$ or 1. This is a principal series representation $\pi(\chi_1, \chi_2)$ where χ_1 and χ_2 are characters of $\mathbb{R}^\times$. (Such a representation is irreducible provided $\chi_1\chi_2^{-1}$ is not of the form $\mathrm{sign}(y)^\epsilon |y|^{n-1}$ where $n \equiv \epsilon \mod 2$.)
- Otherwise V is a discrete series or limit discrete series. In this case, there is a positive integer k such that $V(l) \neq 0$ precisely when $l = \pm(k + 2n)$ with $n \in \mathbb{Z}_{\geq 0}$.

  Concretely, the discrete series can be realized as the K-finite vectors in the representation of $\mathrm{GL}_2(\mathbb{R})^+$ on the space of holomorphic functions $f : \mathcal{H} \to \mathbb{C}$ such that
$$\int_\mathcal{H} |f(z)|^2 y^k \frac{dxdy}{y^2} < \infty.$$
  The action is given by sending f to $f|_k(g^{-1})$.

Remark 3.4. This suggests the discrete series are connected with classical modular forms. We will see they are the component at infinity for the automorphic representation associated to a modular form.

3.3. **Hecke Algebra.** We will give a modern definition of the Hecke algebra in this context: this is slightly different approach than in [JL70, Section 5]. A summary is found in [Bum97, Section 3.4] with references given to Knapp and Vogan [KV95]. One definition of the Hecke algebra $\mathcal{H}_G$ is as the algebra of compactly supported distributions on $\mathrm{GL}_2(\mathbb{R})$ that are supported on K and are K-finite under left and right translation. However, there is a description more closely connected to $(\mathfrak{g}, K)$-modules.

Let $\mathcal{H}_K$ denote the space of smooth functions on K which are K-finite under left and right translation by K. These can also be viewed as distributions by integrating over K. The universal enveloping algebra $U(\mathfrak{gl}_2(\mathbb{C}))$ may be identified with distributions supported at the identity by having $U(\mathfrak{gl}_2(\mathbb{C}))$ act on functions and then evaluate at the identity. For $X \in \mathfrak{gl}_2(\mathbb{C})$, this gives the distribution

$$f \mapsto \frac{d}{dt} f(\exp(tX))|_{t=0}.$$

We record two important facts.

Proposition 3.5. *The natural map $\mathcal{H}_K \otimes_{U(\mathrm{Lie}(K))} U(\mathfrak{gl}_2(\mathbb{C})) \to \mathcal{H}_G$ is an isomorphism.*

Proposition 3.6. *Let V be a $(\mathfrak{g}, K)$-module. Then V is a smooth module for $\mathcal{H}_G$, and every smooth module arises in this way.*

4. Background at the Non-Archimedean Places

In this section, we review classification of irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, and the theory of spherical representations.

4.1. Representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and Hecke Algebras.

Recall that a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on V is admissible if it is smooth and if for every compact open subgroup $K' \subset \mathrm{GL}_2(\mathbb{Q}_p)$ we have $\dim V^{K'} < \infty$. There is a classification of irreducible admissible representations, found in [Bum97, Chapter 4] or [JL70, Sections 2-4] or [BH06, Section 9].

- All finite dimensional irreducible admissible representations are one dimensional, and factor through the determinant.
- There are principal series representations $\pi(\chi_1, \chi_2)$ where χ_1 and χ_2 are characters of $\mathbb{Q}_p^\times$. When $\chi_1\chi_2^{-1} \neq |\cdot|_p^{\pm 1}$, it is irreducible.
- There are special representations, which are irreducible infinite dimensional subquotient of $\pi(\chi_1, \chi_2)$. These are all twists of the Steinberg representation, which occurs for $\chi_1 = |\cdot|_p^{1/2}$ and $\chi_2 = |\cdot|_p^{-1/2}$.
- There are super-cuspidal representations.

The Hecke algebra $\mathcal{H}(G)$ for $G = \mathrm{GL}_2(\mathbb{Q}_p)$ is the convolution algebra of locally constant compactly supported functions. There is a natural action of $\mathcal{H}(G)$ on representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ given by

$$\pi(\phi)v = \int_G \phi(g)\pi(g)v dg.$$

Smooth representations of G are the same as $\mathcal{H}(G)$-modules.

Recall that idempotent elements of $\mathcal{H}(G)$ are given by normalized characteristic functions of compact open subgroups K. The characteristic function of $K = \mathrm{GL}_2(\mathbb{Z}_p)$ is an idempotent for the usual measure.

Definition 4.1. The spherical Hecke algebra $\mathcal{H}(G, K)$ is defined to be $1_K\mathcal{H}(G)1_K$, the space left and right K-invariant elements of $\mathcal{H}(G)$. An smooth admissible representation (π, V) is called spherical (or unramified) if $V^K \neq 0$. A non-zero element is called a spherical vector.

The spherical vectors are a $\mathcal{H}(G, K)$-module. We sketch some of the key facts about spherical vectors as these concepts were not included in [JL70],

Let T_p and R_p be the characteristic functions of $K \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K$ and $K \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} K$.

Theorem 4.2. *The spherical Hecke algebra $\mathcal{H}(G, K)$ is commutative. In particular, it is a polynomial algebra generated by T_p, R_p, and R_p^{-1}.*

Proof. Using the transpose map, one defines an anti-involution of the spherical Hecke algebra. Then one checks that a basis for $\mathcal{H}(G, K)$, given by characteristic functions for double cosets of K, are invariant under transpose [Bum97, Theorem 4.6.1]. Using the p-adic Cartan decomposition, one can describe all double cosets for K. Decomposing them as a union of left cosets, one can build all of them out of T_p, R_p, and R_p^{-1}. For details, see Propositions 4.6.4 and 4.6.5 in [Bum97]. $\square$

Remark 4.3. The identification of $\mathcal{H}(G, K)$ with the polynomial algebra is known as the Satake isomorphism.

The other key fact says that the spherical vectors determine the representation.

Theorem 4.4. *For an irreducible unramified representations (π, V), V^K is one dimensional. There is an equivalence of categories between irreducible unramified representations and irreducible $\mathcal{H}(G, K)$-modules sending V to V^K.*

Proof. It is easy to see that V^K is an irreducible $\mathcal{H}(G, K)$-module, and it is automatically finite dimensional. As $\mathcal{H}(G, K)$ is commutative, it is one dimensional. [Bum97, Theorem 4.6.2]

A quasi-inverse is defined in [BH06, 4.3 Proposition]. It sends a $\mathcal{H}(G, K)$-module M to an irreducible submodule of $U = \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M$. The key content is using Zorn's lemma to construct a maximal subspace $X \subset U$ for which $X^K = 0$. Anything outside X must generate U together with X. Hence the quotient U/X is an irreducible submodule with $(U/X)^K = M$. One checks this is an equivalence of categories. $\square$

Thus the way T_p and R_p act completely determines an unramified representation. It is convenient to record their action using Satake parameters: if T_p and R_p act by λ and μ, the Satake parameters are the roots of $x^2 - p^{k/2-1}\lambda x + \mu p^{k-1}$.

4.2. **Classification of Irreducible Spherical Representations.** We now analyze which irreducible representations are spherical.

If (π, V) is one dimensional, then $\mathrm{GL}_2(\mathbb{Q}_p)$ acts by $\chi \circ \det$ where χ is a quasi-character of $\mathbb{Q}_p^\times$. This is spherical if χ is unramified: in other words, if χ is trivial on $\mathbb{Z}_p^\times$.

The principal series $\pi(\chi_1, \chi_2)$ can be realized as the space of smooth functions $f : G \to \mathbb{C}$ such that

$$f(bg) = \chi_1 \otimes \chi_2(b)\delta(b)^{1/2} f(g)$$

where $b = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is in the standard Borel B. Here $\chi_1 \otimes \chi_2(b) = \chi_1(a)\chi_2(d)$ and $\delta(b) = |a/d|$ is the modular character. The Iwasawa decomposition gives that $G = BK$, and so a candidate for a spherical function is

$$f_{sph}(bk) = \chi_1 \otimes \chi_2(b)\delta(b)^{1/2}.$$

This function is well defined provided it is trivial on $K \cap B$. All such matrices are upper triangular and have units on the diagonal, so provided χ_1 and χ_2 are unramified (trivial on $\mathbb{Z}_p^\times$), f_{sph} is well defined and gives a spherical vector.

To determine the $\mathcal{H}(G, K)$ action on $\pi(\chi_1, \chi_2)^K$, we just need to understand how T_p and R_p act. The double coset for T_p decomposes as

$$K \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K = \bigcup_{b=0}^{p-1} \begin{pmatrix} p & b \\ 0 & 1 \end{pmatrix} K \cup \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K$$

Then we expand

$$\begin{aligned}
T_p f_{sph} &= \int_K \sum_b^{p-1} f_{sph}\left(\begin{pmatrix} p & b \\ 0 & 1 \end{pmatrix} g\right) + f_{sph}\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} g\right) dg \\
&= f_{sph}\left(\begin{pmatrix} p & b \\ 0 & 1 \end{pmatrix}\right) + f_{sph}\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right) \\
&= p \cdot \chi_1(p)|p|^{1/2} + \chi_2(p)|p|^{-1/2} \\
&= p^{1/2}(\chi_1(p) + \chi_2(p)).
\end{aligned}$$

The double coset for R_p is the single coset $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} K$, so

$$\begin{aligned}
R_p f_{sph} &= \int_K f_{sph}\left(\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} g\right) dg \\
&= f_{sph}\left(\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\right) \\
&= \chi_1(p)\chi_2(p).
\end{aligned}$$

Proposition 4.5. *The only spherical representations are $\chi \circ \det$ with χ unramified, and irreducible $\pi(\chi_1, \chi_2)$ with χ_1 and χ_2 unramified.*

Proof. Let (V, π) be spherical. Let T_p act on V^K by λ and R_p act by μ. Note that μ is invertible as R_p is invertible in the Hecke algebra. The Satake parameters α_1 and α_2 are the roots of the quadratic $x^2 - p^{k/2-1}\lambda x + \mu p^{k-1}$. Let χ_1 and χ_2 be the unramified quasi-characters such that $\chi_1(p) = \frac{\alpha_1}{p^{k-1/2}}$ and $\chi_2(p) = \frac{\alpha_2}{p^{k-1/2}}$. If $\pi(\chi_1, \chi_2)$ is irreducible, its K-fixed vectors have the same Satake parameters. This forces $V \simeq \pi(\chi_1, \chi_2)$.

So the only remaining possible case is when $\pi(\chi_1, \chi_2)$ is not irreducible. This happens only when $\alpha_1\alpha_2^{-1} = p^{\pm 1}$. Without loss of generality, we may assume that $\alpha_1\alpha_2^{-1} = p$, so by our knowledge of the principal series we know $\pi(\chi_1, \chi_2)$ has a one dimensional invariant subspace. But then this must be the K-fixed vectors, so the representation is one dimensional. $\square$

Remark 4.6. The conductor measures how far a representation is from being spherical. Let $K_0(c)$ denote matrices in $\mathrm{GL}_2(\mathbb{Z}_p)$ with lower left entry a multiple of p^c, and let (π, V) be an infinite dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with central character ω. Then there exists a $c \geq 0$ such that

$$\{v \in V : \pi(g)v = \omega(g)v \,\forall g \in K_0(c)\}$$

is non-zero by a theorem of Casselman. The smallest such c for which this holds is called the conductor of π. In that case, the space is one dimensional. A non-zero element is called a new vector.

For the principal series and Steinberg representation, it is relatively simple to find a new vector and compute the conductor by generalizing the construction of a spherical vector [Sch02]. In the case of the irreducible principal series $\pi(\chi_1, \chi_2)$, the conductor is the sum of the conductors of χ_1 and χ_2. For the Steinberg, the conductor is 1. For a supercuspidal representation, the conductor is least two. Furthermore, the central character of a supercuspidal has conductor at most $[\frac{r}{2}]$ (see [AL78, Theorem 4.3']).

5. The Automorphic Representation Associated to a Modular Form

Let f be a holomorphic modular form of level N and nebentypus character character χ. Suppose further that it is a cusp form and an eigenfunction for the Hecke operators T_p for $p \nmid N$. We have associated an adelic automorphic form $\varphi_f \in \mathcal{A}_0(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}), \omega)$. In this section, we will prove the following result:

Theorem 5.1. *The automorphic form φ_f lies in an unique irreducible admissible automorphic representation $\pi_f \subset \mathcal{A}_0(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}), \omega)$.*

The representation π_f can be expressed as a restricted tensor product $\otimes'_v \pi_{f,v}$. We would also like to describe the components $\pi_{f,v}$ in terms of the properties of f. After reviewing some facts about the Hecke algebra and connecting it to the classical Hecke operators, we state the strong multiplicity one theorem and then turn to proving Theorem 5.1 and discussing the local components.

5.1. **The Hecke Algebra and Hecke Operators.** The global Hecke algebra $\mathcal{H}_{\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})}$ is defined to be the restricted tensor product of the local Hecke algebras for the archimedean and non-archimedean places of $\mathbb{Q}$ reviewed in the previous two sections. For this to be defined, we needed to specify a spherical idempotent in all but finitely many of the local Hecke algebras. We use the characteristic function of $\mathrm{GL}_2(\mathbb{Z}_p)$ in $\mathcal{H}_{\mathrm{GL}_2(\mathbb{Q}_p)}$.

The representation π_f can be viewed as a $\mathcal{H}_{\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})}$-module. There is a factorization $\pi_f = \otimes'_v \pi_{f,v}$ where $\pi_{f,v}$ is a $\mathcal{H}_{\mathrm{GL}_2(\mathbb{Q}_v)}$-module. This also means that there is a spherical vector in almost all of the $\pi_{f,v}$ (a vector invariant under the action of $\mathrm{GL}_2(\mathbb{Z}_v)$). Such vectors are unique up to scaling.

The first step is to connect the classical Hecke operators with the adelic ones. We studied the Hecke operator $T_p \in \mathcal{H}_{\mathrm{GL}_2(\mathbb{Q}_p)}$ given by convolution with the characteristic function of $H_p = \mathrm{GL}_2(\mathbb{Z}_p)\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)$ in Section 4. In the following proposition, $\begin{pmatrix} a & b \\ c & d\end{pmatrix}_p$ will denote an element of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with specified matrix at p and the identity elsewhere.

Proposition 5.2. *Let f be a modular form of level N, and suppose p is a prime such that $p \nmid N$. Then $T_p(\varphi_f) = \varphi_{p^{1-k/2}T_p f}$.*

Proof. It suffices to check equality for $g = g_\infty \in \mathrm{GL}_2(\mathbb{R})^+$ because φ_f is left $\mathrm{GL}_2(\mathbb{Q})$-invariant and because of Remark 2.4.

Recall that the double coset decomposes as

$$H_p = \bigcup_{b=0}^{p-1} \begin{pmatrix} p & b \\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p) \cup \begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)$$

We calculate that

$$T_p(\varphi_f)(g) = \sum_{b=0}^{p-1}\int_{\mathbb{Z}_p}\varphi_f\left(g\begin{pmatrix} p & b \\ 0 & 1\end{pmatrix}_p k\right)dk + \int_{\mathbb{Z}_p}\varphi_f\left(g\begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}_p k\right)dk.$$

As $p \nmid N$, the p component of $K_0(N)$ is $\mathrm{GL}_2(\mathbb{Z}_p)$ and λ is trivial on it (Lemma 2.2). So by Remark 2.4, φ_f is right invariant under $\mathrm{GL}_2(\mathbb{Z}_p)$. Hence

$$\int_{\mathbb{Z}_p}\varphi_f\left(g\begin{pmatrix} p & b \\ 0 & 1\end{pmatrix}_p k\right)dk = \varphi_f\left(g\begin{pmatrix} p & b \\ 0 & 1\end{pmatrix}_p\right)$$

because the volume of $\mathbb{Z}_p$ is one. There is a similar expression for the other coset. Therefore we unwind to see that

$$T_p(\varphi_f)(g) = \sum_{b=0}^{p-1}\varphi_f\left(g\begin{pmatrix} p & b \\ 0 & 1\end{pmatrix}_p\right) + \varphi_f\left(g\begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}_p\right).$$

Now let $z = g_\infty \cdot i$, and note that

$$g_\infty\begin{pmatrix} p & b \\ 0 & 1\end{pmatrix}_p = \begin{pmatrix} p & b \\ 0 & 1\end{pmatrix}_\mathbb{Q}\left(\begin{pmatrix} p & b \\ 0 & 1\end{pmatrix}_\infty^{-1} g_\infty\right)\alpha$$

where $\alpha_v = \begin{pmatrix} p & b \\ 0 & 1 \end{pmatrix}^{-1}$ for $v \neq p, \infty$ and is the identity otherwise. Observe that $\lambda(\alpha) = 1$. Therefore

$$\varphi_f\left(g_\infty \begin{pmatrix} p & b \\ 0 & 1 \end{pmatrix}_p\right) = F\left(\begin{pmatrix} p & b \\ 0 & 1 \end{pmatrix}_\infty^{-1} g_\infty\right) = p^{-k/2} \det(g_\infty)^{k/2} j(g_\infty, i)^{-k} f\left(\frac{z-b}{p}\right).$$

Doing the same for the remaining coset gives

$$\varphi_f\left(g_\infty \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right) = p^{k/2} \det(g_\infty)^{k/2} j(g_\infty, i)^{-k} f\left(pz\right).$$

Therefore we conclude that

$$\begin{aligned} p^{k/2-1} T_p(\varphi_f)(g) &= \left(\sum_{b=0}^{p-1} \frac{1}{p} \det(g_\infty)^{k/2} j(g_\infty, i)^{-k} f\left(\frac{z-b}{p}\right)\right) + p^{k-1} \det(g_\infty)^{k/2} j(g_\infty, i)^{-k} f(pz) \\ &= \det(g_\infty)^{k/2} j(g_\infty, i)^{-k} (T_p f)(z) \\ &= \varphi_{T_p f}(g_\infty) \end{aligned}$$

using the standard normalizations of the classical Hecke operators. □

Similarly, we interpret the operator R_p given by convolution with the characteristic function of $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p)$ to

Proposition 5.3. *For $p \nmid N$, we have $R_p \varphi_f = \chi(d) \varphi_f$.*

Proof. We simplify

$$(R_p \varphi_f)(g_\infty) = \int_{\mathrm{GL}_2(\mathbb{Z}_p)} \varphi_f\left(g_\infty \begin{pmatrix} p & 0 \\ p & 0 \end{pmatrix}_p k\right) dk = \varphi_f\left(g_\infty \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}_p\right).$$

But we can rearrange

$$g_\infty \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}_p = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}_{\mathbb{Q}} \left(\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}_\infty^{-1} g_\infty\right) \alpha$$

where $\alpha_v = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1}$ for $v \neq p, \infty$ and is the identity elsewhere. We see that

$$\varphi_f\left(g_\infty \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}_p\right) = F\left(\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1} g_\infty\right) \lambda(\alpha).$$

Now $\lambda(\alpha) = \chi(p)$ by Lemma 2.2, and F is left invariant under $Z(\mathbb{R})^+$. Thus

$$R_p \varphi_f = \chi(p) \varphi_f.$$ □

5.2. **Proof of Theorem 5.1.** The proof will use the multiplicity one theorem, a proof of which may be found in [JL70, Section 11] or [Bum97, Section 3.5].

Theorem 5.4 (Multiplicity One)**.** *Let (π, V) and (π', V') be two irreducible admissible subrepresentations of $\mathcal{A}_0(\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}), \omega)$. Assume that $\pi_v \simeq \pi'_v$ for all but finitely many places. Then $V = V'$.*

Using this, we now prove Theorem 5.1. Let f be a cusp form and Hecke eigenform. By the general theory, the space of automorphic forms decomposes as a direct sum. Let (π, V) be one irreducible factor such that the projection of φ_f is non-zero. We will show that all of the local components of π at places not dividing N or infinity are determined by f. Let $p \nmid N$ be prime. We know that φ_f is invariant under right translation by $K_0(N)$, so in particular the local component π_p contains an element which is left and right $\mathrm{GL}_2(\mathbb{Z}_p)$-translation invariant. It is an eigenvector for T_p and R_p with eigenvalues determined by f, and this action determines the action of $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$. In other words, the local component π_p is an spherical representation of $\mathcal{H}_p$ that is completely specified in terms of f. By the multiplicity one theorem, this forces φ_f to lie in a unique irreducible (π, V), proving the theorem.

Remark 5.5. It is also easy to analyze the component at infinity. The version of the multiplicity one statement proven in Bump is slightly weaker in that it needs an isomorphism at the archimedean places, so it is worth spelling this out.

From Section 3, we know quite a bit about the potential $(\mathfrak{g}, K)$-modules at infinity. In particular, the formula for the lowering operator in Remark 3.2 allows us to calculate that for $F(g) = e^{-ik\theta} y^{k/2} f(x+iy)$ we have

$$\begin{aligned} L \cdot \varphi_f &= e^{2i\theta}\left(-iy\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + \frac{1}{2i}\frac{\partial F}{\partial \theta}\right) \\ &= e^{2\pi\theta}\left(-ie^{-ik\theta}y^{k/2+1}f_x + e^{-ik\theta}\frac{k}{2}y^{k/2}f + e^{-ik\theta}y^{k/2+1}f_y + \frac{1}{2i}(-ik)e^{-ik\theta}y^{k/2}f\right) \\ &= 0 \end{aligned}$$

where the last step uses that f is holomorphic so $f_y = if_x$. So any irreducible automorphic representation to which φ_f projects non-trivially contains a vector on which L acts by zero. We already know that Δ acts by $\frac{k}{2}\left(1 - \frac{k}{2}\right)$. Using the classification of irreducible $(\mathfrak{g}, K)$-modules, we see the only one with this property is the discrete series of weight k. This should not be a surprise, as one realization of this representation includes a holomorphic modular form.

It is harder to characterize the local component $\pi_{f,p}$ for $p|N$. A recent paper by Loeffler and Weinstein gives an algorithm for computing the local components [LW12]. The hard case is when the local component is supercuspidal. We sketch a description of the non-supercuspidal cases using the concept of conductor from Remark 4.6.

We say that $\pi_{f,p}$ is p-primitive if the conductor is minimal among the conductors of the p-components of twists of f by characters. An irreducible principal series $\pi(\chi_1, \chi_2)$ is primitive if at least one of the characters is unramified. The conductor is the sum of the conductors for χ_1 and χ_2, so if both were unramified we could twist to reduce the conductor. A special representation is a twist of a Steinberg, and the unramified twists the Steinberg representation have minimal conductor (one).

So suppose $\pi_{f,p}$ is p-primitive, and let p^r be the largest power of p which divides N. We know that r is the conductor of $\pi_{f,p}$. Recall ω_p is the p-component of the central character, and was defined so that it is trivial provided χ factors through $\mathbb{Z}/(N/p^r)\mathbb{Z}^\times$.

We know that the central character of a supercuspidal representation has conductor at most $[\frac{r}{2}]$. Furthermore, we know supercuspidals must have conductor at least 2. So if ω_p has conductor more than $[r/2]$, π_p must be a principal series representation. The only p-primitive one is $\pi(\chi_1, \chi_2)$ with χ_1 having conductor r and with χ_2 unramified. If $r = 1$ and ω_p is unramified, π_p must be a twist of a Steinberg by an unramified character. The remaining cases must then be supercuspidal.

The conclusion of this analysis is the following theorem.

Theorem 5.6. *Let* $f(z) = \sum_{n \geq 1} a_n q^n$ *be a cuspidal eigenform of weight* k*, level* N *and nebentypus character* χ*. Let* π_f *be the associated irreducible automorphic representation, with* $\pi_f = \otimes_v' \pi_{f,v}$*. For a prime* $p|N$*,* ω_p *is the* p*-component of the central character.*

- *For* $v = \infty$*, the local component* $\pi_{f,\infty}$ *is a discrete series of weight* k*.*

- *For $p \nmid N$, the local component $\pi_{f,p}$ is an unramified principal series. On the spherical vector, T_p and R_p act via multiplication by $a_p p^{1-k/2}$ and $\chi(p)$. The Satake parameters are therefore the roots of $x^2 - a_p x + \chi(p)p^{k-1}$.*
- *Suppose $N = p^r N'$ where $p \nmid N'$ and that f is a new form at p and that f is p-primitive. If the conductor of $\pi_{f,p}$ is p^r, the local factor is the principal series $\pi(\chi_1, \chi_2)$ where χ_1 is unramified and satisfies $\chi_1(p) = a_p/p^{(k-1)/2}$ and $\chi_1\chi_2 = \omega_p$.*
- *With the same notation, if $r = 1$ and ω_p is unramified, then the local factor $\pi_{f,p}$ is the twist of the Steinberg representation by an unramified character χ_1 such that $\chi_1(p)p^{(k-2)/2} = a_p$.*
- *Otherwise the local component at p is supercuspidal, and can be described by the algorithm in [LW12].*

An example of the automorphic representation associated to a weight 2 level 99 modular form is included in the introduction.

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