BRAUER GROUPS: TALK 1

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In this first talk, we will discuss the topological Brauer group and the Brauer group of $\operatorname{Spec} k$, which are classically studied in the guise of central simple algebras. Most of this material is presented in Grothendieck's Groupe de Brauer I, found in [2]. Details about central simple algebras are drawn from Chapter IV of Milne's notes on Class Field Theory [3]. This presentation is partly inspired by notes from a talk by Pete Clark [1].

1. The Brauer Group of a Topological Space

We first study the Brauer group of a topological space, which can be interpreted in many ways: as classifying projective bundles up to projectivizations of vector bundles, as a quotient of the union of the cohomology groups $H^1(X, \mathrm{PGL}_n)$, as classifying Azumaya algebras up to endomorphism algebras of vector bundles, or as the torsion in $H^3(X, \mathbb{Z})$. We will prove these equivalences, which are a model for the subsequent discussion of the Brauer group in algebraic geometry.

1.1. **Projective Bundles.**

Just as K-theory classifies vector bundles on a space, the Brauer group will help classify projective bundles. This is not the first place Brauer groups appeared historically, but will be a convenient place to begin. For now, let X simply be a topological space, and consider $\mathbb{CP}^n$ bundles, by which we mean a fiber bundle with fiber $\mathbb{CP}^n$ and structure group $\mathrm{Aut}(\mathbb{CP}^n) = \mathrm{PGL}_{n+1}(\mathbb{C})$. If a bundle is trivialized on a cover $\{U_\alpha\}$ of X, then to specify the bundle all we need to do is specify transition functions on all the $U_\alpha \cap U_\beta$ between the two trivializations, subject to the constraint that they are compatible with the triple overlaps. But this gives a 1-cocycle, with values in the structure group $\mathrm{PGL}_{n+1}(\mathbb{C})$. As usual, changing the 1-cocycle by a coboundary yields an isomorphic bundle. Therefore $H^1(X, \mathrm{PGL}_{n+1}(\mathbb{C}))$ classifies $\mathbb{CP}^n$ bundles on X.

Note that since $\mathrm{PGL}_{n+1}(\mathbb{C})$ is a non-Abelian group, this is not ordinary sheaf cohomology. We can still define H^1 via cocycles, but cannot define the usual H^n because the non-commutativity interferes with the required calculations with cocycles, and prevents appealing to more abstract machinery. This is similar to the situation in group cohomology, where $H^1(G, M)$ is defined for non-Abelian G-modules via cocycles, but the higher ones are not.

There is one obvious class of projective bundles on X, those which we obtain by projectivizing a complex vector bundle. Since $H^1(X, \mathrm{GL}_{n+1}(\mathbb{C}))$ classifies rank $n+1$ vector bundles, again via transition functions, the natural map on cocycles $H^1(X, \mathrm{GL}_{n+1}(\mathbb{C})) \to H^1(X, \mathrm{PGL}_{n+1}(\mathbb{C}))$ corresponds to projectivization. We are most interested then in $H^1(X, \mathrm{PGL}_{n+1}(\mathbb{C}))$ modulo the image, although we do not yet have a group structure. We expect there to be one by analogy with the Picard group and K-theory. This is easiest to do by reinterpreting the Brauer group.

1.2. **Azumaya Algebras.**

By the Noether-Skolem theorem (Theorem 15 here), $\mathrm{PGL}_n(\mathbb{C})$ is the group of automorphisms of $\mathrm{Mat}_n(\mathbb{C})$ since all automorphisms are inner. So $H^1(X, \mathrm{PGL}_n(\mathbb{C}))$ can also be interpreted as giving the transition functions for a sheaf of $\mathcal{O}_X$ algebras that is locally isomorphic to $\mathrm{Mat}_n(\mathbb{C})$. This leads to the study of Azumaya algebras.

Definition 1. An Azumaya algebra is an $\mathcal{O}_X$ algebra that is locally isomorphic to the sheaf $\mathrm{Mat}_n(\mathbb{C})$ on X for some n.

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Remark 2. It is equivalent to require that an Azumaya algebra be a finite, locally free $\mathcal{O}_X$ module A such that $A \otimes \mathcal{O}_{X,x} \simeq \mathrm{Mat}_n(\mathbb{C})$ for every point $x \in X$.

By using cocycles, one can pass between Azumaya algebras and projective bundles.

As with the projective bundles, there is a class of Azumaya algebras which are boring: those which arise as the endomorphisms of a vector bundle. These again are precisely the cocycles in the image of $H^1(X, \mathrm{GL}_n(\mathbb{C}))$ in $H^1(X, \mathrm{PGL}_n(\mathbb{C}))$. The Brauer group can then be interpreted as Azumaya algebras up to endomorphisms of a vector bundle. So two Azumaya algebras A_1 and A_2 are equivalent provided there are vector bundles E_1 and E_2 on X so that $A_1 \otimes \mathrm{End}(E_1) \simeq A_2 \otimes \mathrm{End}(E_2)$.

Using Azumaya algebras, it is straightforward to define a group structure on $\mathrm{Br}(X)$. Given Azumaya algebras A_1 and A_2 on X, their product is the tensor product $A_1 \otimes A_2$ which is obviously a Azumaya algebra (of different rank). The inverse of A_1 is the opposite Azumaya algebra A_1^{opp}, since we may check locally and we know that if A is a matrix algebra then there is an isomorphism $A \otimes A^{\mathrm{opp}} \simeq \mathrm{End}(A)$. Since $\mathrm{End}(A_1 \otimes A_2) \simeq \mathrm{End}(A_1) \otimes \mathrm{End}(A_2)$, the product respects equivalence of Azumaya algebras.

Definition 3. The Brauer group of X, denoted by $\mathrm{Br}(X)$, is the group of Azumaya algebra up to equivalence.

Using the discussion in terms of cocycles, the Brauer group can be interpreted as the union of the $H^1(X, \mathrm{PGL}_{n+1}(\mathbb{C}))$ as n varies modulo the images of $H^1(X, \mathrm{GL}_{n+1}(\mathbb{C}))$.

1.3. **Computations of the Brauer Group of a Topological Space.** There are several useful representations of the Brauer group of X besides projective bundles and Azumaya algebras.

Proposition 4. *Let X be a topological space.* $\mathrm{Br}(X)$ *is an Abelian torsion subgroup of* $H^2(X, \mathcal{O}_X^\times)$.

Theorem 5. *Let X be a finite CW complex. Then* $\mathrm{Br}(X) \simeq H^2(X, \mathcal{O}_X^\times)[tors] \simeq H^3(X, \mathbb{Z})[tors]$.

The first statement is elementary, but the second requires some significant topology and is due to Serre. To be clear, as X is simply a topological space $\mathcal{O}_X$ is the sheaf of continuous complex valued functions on X.

Proof. From the short exact sequence of sheaves

$$1 \to \mathcal{O}_X^\times \to \mathrm{GL}_n(\mathbb{C})_X \to \mathrm{PGL}_n(\mathbb{C})_X \to 1$$

we obtain an exact sequence of non-Abelian cohomology

$$H^1(X, \mathrm{GL}_n(\mathbb{C})) \to H^1(X, \mathrm{PGL}_n(\mathbb{C})) \xrightarrow{\delta} H^2(X, \mathcal{O}_X^\times)$$

The connecting homomorphism δ gives a map from Azumaya algebras that are locally $\mathrm{Mat}_n(\mathbb{C})$ to $H^2(X, \mathcal{O}_X^\times)$. It is injective since the kernel is $H^1(X, \mathrm{GL}_n(\mathbb{C}))$, which correspond to the trivial Azumaya algebras which are endomorphisms of vector bundles. The fact that $\delta(A_1) \cdot \delta(A_2) = \delta(A_1 \otimes A_2)$ can be verified by a computation with cocycles. By piecing together the maps for all n, this shows that $\mathrm{Br}(X)$ injects into $H^2(X, \mathcal{O}_X^\times)$ and that the resulting map is a homomorphism.

It remains to show that $\mathrm{Br}(X)$ is torsion. Any element comes from some $H^1(X, \mathrm{PGL}_n(\mathbb{C}))$ for some n. Consider the pair of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}_X^\times \arrow[r] & \mathrm{GL}_n(\mathbb{C})_X \arrow[r] & \mathrm{PGL}_n(\mathbb{C})_X \arrow[r] & 1 \\
1 \arrow[r] \arrow[u] & \mu_n \arrow[r] \arrow[u] & \mathrm{SL}_n(\mathbb{C})_X \arrow[r] \arrow[u] & \mathrm{PGL}_n(\mathbb{C})_X \arrow[r] \arrow[u] & 1 \arrow[u]
\end{tikzcd}$$

and the corresponding exact sequences of cohomology

$$\begin{array}{ccccc} H^1(X,\mathrm{GL}_n(\mathbb{C})) & \longrightarrow & H^1(X,\mathrm{PGL}_n(\mathbb{C})) & \longrightarrow & H^2(X,\mathcal{O}_X^\times) \\ \uparrow & & \uparrow & & \uparrow \\ H^1(X,\mathrm{SL}_n(\mathbb{C})) & \longrightarrow & H^1(X,\mathrm{PGL}_n(\mathbb{C})) & \longrightarrow & H^2(X,\mu_n) \end{array}$$

Since $H^2(X,\mu_n)$ is n-torsion and the middle map is an isomorphism, the image of $H^1(X,\mathrm{PGL}_n(\mathbb{C}))$ in $H^2(X,\mathcal{O}_X^\times)$ is n-torsion. □

To prove the theorem, we first need a consequence of the exponential exact sequence that has nothing to do with Brauer groups.

Lemma 6. *Let X be a paracompact topological space. Then $H^2(X,\mathcal{O}_X^\times) \simeq H^3(X,\mathbb{Z})$.*

Proof. We consider the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 1.$$

Since $\mathcal{O}_X$ is acyclic, part of the long exact sequence of cohomology reads

$$0 \to H^2(X,\mathcal{O}_X^\times) \to H^3(X,\mathbb{Z}) \to 0.$$ □

Therefore we can just work with $H^3(X,\mathbb{Z})$, and attempt to show its torsion is the Brauer group. We rely on the following fact, which we will sketch a proof of at the end: the classifying space $\mathrm{BPGL}_\infty(\mathbb{C})$ is homotopy equivalent to $K(\mathbb{Q}/\mathbb{Z},2) \times K(\mathbb{Q},4) \times K(\mathbb{Q},6) \times \ldots$.

Assuming this, given a torsion element $x \in H^3(X,\mathbb{Z})$, the long exact sequence of cohomology coming from $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ over X reads

$$H^2(X,\mathbb{Q}) \to H^2(X,\mathbb{Q}/\mathbb{Z}) \xrightarrow{d} H^3(X,\mathbb{Z}) \to H^3(X,\mathbb{Q}).$$

Since $H^3(X,\mathbb{Q})$ is torsion free, the image of x is zero, so x is dy for some $y \in H^2(X,\mathbb{Q}/\mathbb{Z})$. But an element of this cohomology class is equivalent to a map from X to $K(\mathbb{Q}/\mathbb{Z},2)$ up to homotopy. Since $K(\mathbb{Q}/\mathbb{Z},2)$ is a factor of a product homotopy equivalent to $\mathrm{BPGL}_\infty(\mathbb{C})$, this gives a map from X to $\mathrm{BPGL}_\infty(\mathbb{C})$, which corresponds to a $\mathrm{PGL}_n(\mathbb{C})$ bundle on X for some n. This gives a way to construct a bundle given a torsion element of $H^3(X,\mathbb{Z})$, which one checks gives an inverse to the inclusion $\mathrm{Br}(X) \hookrightarrow H^3(X,\mathbb{Z})[tors]$.

Finally, it remains to sketch why the classifying space $\mathrm{BPGL}_\infty(\mathbb{C})$ is homotopy equivalent to $K(\mathbb{Q}/\mathbb{Z},2)\times K(\mathbb{Q},4)\times K(\mathbb{Q},6)\times\ldots$ First, what is $\mathrm{PGL}_\infty(\mathbb{C})$? There are natural maps $\mathrm{PGL}_n(\mathbb{C}) \to \mathrm{PGL}_{nk}(\mathbb{C})$ for $k \geq 1$ which send A to the block diagonal matrix with A in each of the k blocks on the diagonal. (This corresponds to taking the tensor product with the k by k identity matrix.) $\mathrm{PGL}_\infty(\mathbb{C})$ is the limit of this system. We first work with $\mathrm{GL}_\infty(\mathbb{C})$, constructed in an analogous way.[1] Now $\mathrm{PGL}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})/\mathbb{C}^\times$, so we embed $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{nk}(\mathbb{C})$ and consider

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) & \longrightarrow & \mathrm{PGL}_n(\mathbb{C}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathrm{GL}_{nk}(\mathbb{C}) & \longrightarrow & \mathrm{PGL}_n(\mathbb{C}) & \longrightarrow & 1 \end{array}$$

The maps between the diagonally embedded $\mathbb{C}^\times$ are the identity, and in the limit we have

$$1 \to \mathbb{C}^\times \to \mathrm{GL}_\infty(\mathbb{C}) \to \mathrm{PGL}_\infty(\mathbb{C}) \to 1.$$

First, note the maps between the homotopy groups of $\mathbb{C}^\times$ are the identity, so in the limit the only non-trivial homotopy group of the first term is $\pi_1(\mathbb{C}^\times) = \mathbb{Z}$.

[1] This is not the same as including $GL_n(\mathbb{C})$ inside $\mathrm{GL}_{n+1}(\mathbb{C})$ as is often done.

Since $\mathrm{GL}_n(\mathbb{C})$ is homotopy equivalent to $U(n)$, we understand the homotopy groups in the limit by Bott periodicity: the homotopy groups of $U(\infty)$ are 0, $\mathbb{Z}$, 0, $\mathbb{Z}, \ldots$. For fixed k and large n, this means that $\pi_l(U(n))$ is 0 or $\mathbb{Z}$ depending on the parity of l. What are the maps of homotopy groups induced by $U(n) \to U(nk)$? Sending A to a block diagonal matrix with k copies of A on the diagonal can be seen to be homotopy equivalent to sending A to the block diagonal matrix with A^k in one entry and the the identity in the remaining diagonal entries. Therefore the map is multiplication by k, so taking the limit of this system gives $\mathbb{Q}$. Since $\mathrm{GL}_\infty(\mathbb{C})$ has the same homotopy groups of $\Omega\,\mathrm{BGL}_\infty(\mathbb{C})$, we find that $\pi_k(\mathrm{BGL}_\infty(\mathbb{C})) = \mathbb{Q}$ for $k = 2m$, $m \geq 1$, and 0 otherwise.

Finally, we use the exact sequence $1 \to \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C}) \to 1$ in the limit to relate $\mathrm{BPGL}_\infty(\mathbb{C})$ to $\mathrm{BGL}_\infty(\mathbb{C})$. This tells us $\mathrm{BGL}_n(\mathbb{C})$ is a fiber bundle over $\mathrm{BPGL}_n(\mathbb{C})$ with fiber $B\mathbb{C}^\times$. $B\mathbb{C}^\times$ has $\mathbb{Z}$ as its second homotopy group, and no other non-trivial ones. Therefore the long exact sequence of homotopy groups in degree three or higher shows that $\pi_k(\mathrm{BGL}_\infty(\mathbb{C})) = \pi_k(\mathrm{BPGL}_\infty(\mathbb{C}))$. In low degree it reads

$$\ldots \to 0 \to 0 \to \pi_3(\mathrm{BPGL}_\infty(\mathbb{C})) \to \mathbb{Z} \to \mathbb{Q} \to \pi_2(\mathrm{BPGL}_\infty(\mathbb{C})) \to 0.$$

As the map $\mathbb{Z} = \pi_2(B\mathbb{C}^\times) \to \pi_2(\mathrm{BGL}_\infty(\mathbb{C})) = \mathbb{Q}$ is certainly non-zero, $\mathrm{BPGL}_\infty(\mathbb{C})$ has homotopy groups 0, 0, $\mathbb{Q}/\mathbb{Z}$, 0, $\mathbb{Q}$, 0, $\mathbb{Q}$, $0 \ldots$ Finally, an additional argument is needed to show that $\mathrm{BPGL}_\infty(\mathbb{C})$ is homotopy equivalent to a product $K(\mathbb{Q}/\mathbb{Z}, 2) \times K(\mathbb{Q}, 4) \times K(\mathbb{Q}, 6) \ldots$

2. The Brauer Group and Central Simple Algebras

We now turn to the Brauer group of a field, which historically was studied first. However, we still motivate the discussion by thinking about projective bundles on $\operatorname{Spec} k$.

2.1. The Brauer Group in Algebraic Geometry.

In the last section, we used the Brauer group of a topological space X to classify projective bundles. What happens if we try to use a scheme X with the Zariski topology? The difficulty is that defining a projective bundle to be locally trivial with fiber $\mathbb{P}^n$ in the Zariski topology is a bad condition. While for modules being fiberwise free is equivalent to being locally free, this no longer holds for projective bundles. This is clearest in the algebraic description: there is no reason to expect a $R-$algebra A such that $A_\mathfrak{p} \simeq \mathrm{Mat}_n$ for every prime $\mathfrak{p}$ of R to be a matrix algebra over R. The right notion is to require a projective bundle to be étale locally trivial.

Definition 7. A projective bundle P on a scheme X is a morphism $P \to X$ such that for all $x \in X$ there is an étale neighborhood U of x so that the base change of P to U is the projective bundle $\mathbb{P}^n_U \to U$.

When $X = \operatorname{Spec} K$, such a bundle is called a Severi-Brauer variety. This criteria can be reformulated in terms of algebras.

Definition 8. An Azumaya algebra A on X is an $\mathcal{O}_X$-algebra which as a module is locally free of finite rank and such that in an étale neighborhood U of any point $x \in X$, the base change to U is isomorphic to a matrix algebra over U.

We can then define the Brauer group like before, using endomorphism algebras of vector bundles to define an equivalence relation and tensor products to define an operation. As before, the inverse to the monoid operation is given by the opposite algebra.

What happens when we apply this to $X = \operatorname{Spec} K$? Although it seems that projective bundles over a one point space must be trivial, since we are using the étale topology there is actually something interesting going on. An Azumaya algebra over $\operatorname{Spec} K$ is a finite dimensional K-algebra A such that there is a finite separable extension K'/K so that the base change $A_{K'} \simeq \mathrm{Mat}_n(K')$ for some n. These objects were studied in the guise of central simple algebras.

Definition 9. A central simple algebra over a field K is a finite dimensional associative K-algebra that is simple and whose center is exactly K.

Example 10. Let K be a field and K^{al} an algebraic closure. Then $\mathrm{Mat}_n(K)$ is a central simple algebra over K. In some cases (like a finite field) these are the only examples. A non-trivial example is the quaternions $\mathbb{H}$ over $\mathbb{R}$. The complex number $\mathbb{C}$ over $\mathbb{R}$ are not an example, as the center is larger than $\mathbb{R}$.

2.2. **Basic Theory of Central Simple Algebras.** The connection between central simple algebras and Azumaya algebras is given in the following proposition.

Proposition 11. *Let K be a field. The following are equivalent.*

(1) *A is a central simple algebra over K.*
(2) *There exists an n such that $A_{K^{\mathrm{al}}} \simeq \mathrm{Mat}_n(K^{\mathrm{al}})$.*
(3) *There exists an n and a finite separable extension L/K such that $A_L \simeq \mathrm{Mat}_n(L)$.*
(4) *A is of the form $\mathrm{Mat}_r(D)$ where D is a division algebra over K, and K is the center of A.*

This shows that we can interpret Brauer group in terms of central simple algebras: two algebras A_1 and A_2 are said to similar (equivalent in the Brauer group) if $A_1 \otimes_K \mathrm{Mat}_{n_1}(K) \simeq A_2 \otimes_K \mathrm{Mat}_m(K)$ for some n, m: the group operation is tensor product.

Proof. A complete proof of this statement is non-trivial, and is scattered throughout [3, Chapter IV]. Here is a sketch.

Wedderburn's theorem states that every simple algebra is a matrix algebra over a division algebra over K, so (1) and (4) are equivalent. The idea behind Wedderburn's theorem is to pick a simple A module S, and consider the centralizer D of A in $\mathrm{End}_k(S)$. Schur's lemma implies this is a division algebra, and the double centralizer theorem says A equal $\mathrm{End}_D(S)$. But then S is a free module over D, so $\mathrm{End}_D(S)$ is $\mathrm{Mat}_r(D^{\mathrm{opp}})$.

Now define $\mathrm{Br}(L/K)$ to be the the kernel of the map $\mathrm{Br}(K) \to \mathrm{Br}(L)$ given by tensoring with L, so it consists of central simple algebras over K which become matrix algebras after a base change to L. In this case, we say that L splits the algebra. For the implication (1) implies (2), we take $L = K^{\mathrm{al}}$. We know by (4) that $A \otimes_K L$ is $\mathrm{Mat}_n(D)$ for a division ring D over L. But since L is algebraically closed, and any element $\alpha \in D$ generates a finite commutative algebra $L(\alpha)$ over L, we have $D = L$. This shows that (2) holds for any central simple algebra. In fact, a finite extension suffices because the basis elements $\mathrm{Mat}_n(K^{\mathrm{al}})$ generate a finite extension of K. A further analysis shows a separable extension suffices, so (3) follows.

The implication that (3) implies (1) follows by descent: $A_L = \mathrm{Mat}_n(L)$ is a central simple algebra over L, if A were not simple then A_L would be a direct sum as well, and if the center of A was larger than K then the center of A_L would be the base change and hence larger than L as well. □

Remark 12. Because every central simple algebra is of the form $\mathrm{Mat}_n(D)$ for some division algebra D over K, and $\mathrm{Mat}_n(D) \simeq \mathrm{Mat}_n(K) \otimes_k D$, the Brauer group also classifies division algebras over K.

There is also a nice relationship between the maximal subfield of A and the fields that split A.

Proposition 13. *Let A be a central simple algebra over K. Then $[A : K] = r^2$. For a subalgebra L of A, the following are equivalent.*

(1) *L is equal to its centralizer in A.*
(2) *L is degree r over K.*
(3) *L is a maximal commutative K-subalgebra of A.*
(4) *L splits A: there is an isomorphism $A_L \simeq \mathrm{Mat}_r(L)$.*

In this case L may be taken to be étale (finite separable) over K.

We omit the proofs, which again are from [3, Chapter IV]. Finally, we want a criteria for when an extension splits an algebra that does not depend on the field being a subalgebra.

Proposition 14. *Let A be a central simple algebra over K. A field L of finite degree over K splits A if and only there is a central simple algebra B similar to A that contains L and is split by L.*

Finally, we mention the Noether-Skolem theorem, which is an important technical result about maps of central simple algebras.

Theorem 15. *Let A be a simple algebra and B a central simple algebra. For any homomorphisms $f, g : A \to B$, there exists an invertible $b \in B$ such that for all $a \in A$*

$$f(a) = bg(a)b^{-1}.$$

Corollary 16. *The automorphism group of $\mathrm{Mat}_n(K)$ is $\mathrm{PGL}_n(K)$.*

Proof. The Noether-Skolem theorem is true if B is a the matrix algebra $\mathrm{Mat}_n(K)$, since we can interpret the homomorphisms as giving two actions of A on K^n. But the structure theory for simple k-algebras imply A-modules with the same dimension are isomorphic. The isomorphism gives the desired element of B.

In general, we use the matrix algebra $B \otimes_K B^{\mathrm{opp}} \simeq \mathrm{End}_K(B)$. The above argument applies to the algebras $A \otimes_K B^{\mathrm{opp}}$ and $B \otimes_K B^{\mathrm{opp}}$ and homomorphisms $f \otimes 1$ and $g \otimes 1$, so there is an element $b \in B \otimes_K B^{\mathrm{opp}}$ so that

$$(f \otimes 1)(a \otimes b') = b(g \otimes 1)(a \otimes b')b^{-1}$$

for $a \in A$ and $b \in B^{\mathrm{opp}}$. Taking $a = 1$, we see that b lies in the centralizer of $K \otimes_K B^{\mathrm{opp}}$ in $B \otimes_K B^{\mathrm{opp}}$, which is just $B \otimes_K K$ as B is central. Thus $b = b_0 \otimes 1$. If we take $b' = 1$, we see that

$$f(a) \otimes 1 = b_0 g(a) b_0^{-1} \otimes 1.$$

This completes the proof. □

2.3. The Brauer Group and Cohomology.

Just as for a topological space, the Brauer group of a field has an interpretation in terms of cohomology, in this case group cohomology. As usual, the Galois cohomology group $H^2(\mathrm{Gal}(L/K), L^\times)$ will be denoted by $H^2(L/K)$.

Theorem 17. *Let K be a field, and L a finite Galois extension of K. The relative Brauer group $\mathrm{Br}(L/K)$ is isomorphic to $H^2(L/K)$.*

Taking the limit over extensions of K and checking compatibility gives the following.

Corollary 18. *Let K be a field, and K^{sep} separable closure. Then the Brauer group $\mathrm{Br}(K)$ is isomorphic to $H^2(K^{\mathrm{sep}}/K)$.*

We will sketch the direct construction of the isomorphism between $\mathrm{Br}(L/K)$ and $H^2(L/K)$. It is also possible to construct an isomorphism using descent [4, Chapter X].

Let A be a central simple algebra over K, and L an étale subalgebra so that $[A : K] = [L : K]^2$. (This means L splits A and L is its own centralizer in A by Proposition 13.) Applying the Noether-Skolem theorem to $\sigma \in \mathrm{Gal}(L/K)$ and the identity automorphism, we see there are elements $e_\sigma \in A^\times$ such that

$$\sigma a = e_\sigma a e_\sigma^{-1}.$$

If f_σ is another such set of elements, $e_\sigma f_\sigma^{-1}$ centralizes L and hence lies in L. Therefore the e_σ are unique up to scaling by $L^\times$. Now let $\varphi(\sigma, \tau) \in L^\times$ be the element such that

$$e_\sigma e_\tau = \varphi(\sigma, \tau) e_{\sigma\tau}.$$

Then expanding $e_\rho e_\sigma e_\tau$ in terms of φ in two different ways using the associative law shows that

$$\varphi(\rho, \sigma)\varphi(\rho\sigma, \tau) = \varphi(\sigma, \tau)^\rho \varphi(\rho, \sigma\tau)$$

so φ is a 2-cocycle. A calculation shows a different choice of e_σ leads to a cohomologous 2-cocycle, so we have a map from such algebras to $H^2(L/K)$.

To go the other direction, start with a 2-cocycle. Define an algebra structure on the L vector space spanned by e_σ for $\sigma \in \mathrm{Gal}(L/K)$ by requiring

$$\sigma(a)e_\sigma = e_\sigma a \quad \text{and} \quad e_\sigma e_\tau = \varphi(\sigma, \tau)e_{\sigma\tau}.$$

It turns out that this is a central simple algebra over K, which obviously contains L and has degree $[L : K]^2$. A further calculation shows that changing φ by a coboundary produces an isomorphic algebra.

These two processes give an identification between central simple algebras A over K containing a field L such that $[A : K] = [L : K]^2$ with the cohomology group $H^2(L/K)$. The final step is to identify the Brauer group with this collection of central simple algebras. Given two such central simple algebras, if they are similar then they are matrix algebras over the same division algebra. But degree considerations force them to be isomorphic. Conversely, Proposition 14 shows ever central simple algebra split by L is similar to a split one that contains L, and hence by Proposition 13 is of the required form. Finally, some additional algebra checks this bijection is a group homomorphism.

Example 19. With this interpretation, it is possible to give easy proofs of classical theorems of Frobenius and Wedderburn. The first states that the only division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. The second says every finite division algebra is a field.

Any division algebra over $\mathbb{R}$ is a central simple algebra over its center, which must be either $\mathbb{R}$ or $\mathbb{C}$. We know that $\mathrm{Br}(\mathbb{C}) = 0$ as $\mathbb{C}$ is algebraically closed. Using the cohomological interpretation, $\mathrm{Br}(\mathbb{R})$ is $H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times)$. Since $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is cyclic of order 2, we have an explicit description of $H^2(\mathbb{C}/\mathbb{R})$ as the fixed points of complex conjugation modulo norms, which is $\mathbb{R}^\times/\mathbb{R}_{>0} \simeq \mathbb{Z}/2\mathbb{Z}$. The trivial element if $\mathbb{R}$, the non-trivial element is the quaternions.

To show every finite division algebra is a field, we must show that the Brauer group of a finite field k is trivial. It is a standard calculation in Galois cohomology using Hilbert 90 and the Herbrand quotient that $H^2(k^{\mathrm{al}}/k)$ is zero [3, Lemma III 1.4].

References

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