BRAUER GROUPS: TALK 2

JEREMY BOOHER

In this talk we will prove two theorems about the Brauer groups of schemes more interesting than $\operatorname{Spec} k$. This is of course only a taste of the many results found in Grothendieck's Groupe de Brauer [2]. To prove these results we must relate the Brauer group to étale cohomology, which introduces one of the fundamental techniques for studying it. These results can be combined to give a circuitous proof of a fundamental result in local class field theory.

Recall that a ring is Henselian if it is local and Hensel's lemma holds. Local fields are standard examples. The first result is due to Grothendieck.

Theorem 1. *Let R be a Henselian ring with residue field k. Then the natural map $\operatorname{Br}(\operatorname{Spec} R) \to \operatorname{Br}(\operatorname{Spec} k)$ is an isomorphism.*

The second result is due to Auslander and Brumer, who were studying the Brauer group independently from Grothendieck and Azumaya using different techniques.

Theorem 2 (Auslander-Brumer)**.** *Let R be a discrete valuation ring with quotient field K and residue field k. Then*

$$0 \to \operatorname{Br}(\operatorname{Spec} R) \to \operatorname{Br}(\operatorname{Spec} K) \to \operatorname{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \to 0$$

is an exact sequence.

Note that $\operatorname{Hom}(\operatorname{Gal}(k^{\text{sep}}/k), \mathbb{Q}/\mathbb{Z})$ is the dual of $\operatorname{Gal}(k^{\text{sep}}/k)$. It is often denoted $X(G_k)$. Putting these together lets us calculate the Brauer group of a local field.

Corollary 3. *Let K be a local field, a field complete with respect to a discrete valuation with finite residue field k. Then $H^2(\operatorname{Gal}_K, (K^{\text{sep}})^\times) = \operatorname{Br}(K) = \mathbb{Q}/\mathbb{Z}$.*

Proof. Let R be the ring of integers in K. The Auslander-Brumer theorem applies to R, giving a short exact sequence

$$0 \to \operatorname{Br}(\operatorname{Spec} R) \to \operatorname{Br}(\operatorname{Spec} K) \to \operatorname{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \to 0.$$

Since the residue field is finite $\operatorname{Br}(\operatorname{Spec} k) = 0$, since we showed last time that every finite division algebra is a field. Theorem 1 then shows that $\operatorname{Br}(\operatorname{Spec} R) = 0$, so $\operatorname{Br}(\operatorname{Spec} K) = \operatorname{Hom}(G_k, \mathbb{Q}/\mathbb{Z})$. But the absolute Galois group of a finite field is $\hat{\mathbb{Z}}$, so $\operatorname{Br}(\operatorname{Spec} K) = \mathbb{Q}/\mathbb{Z}$. □

1. The Brauer Group and Étale Cohomology

Just as the topological Brauer group is related to $H^2(X, \mathcal{O}_X^\times)$, the Brauer group of a scheme is related to the étale cohomology group $H^2(X_{et}, \mathbb{G}_m)$. However, the relationship is not as tight as in the topological setting. The main result is the following theorem.

Theorem 4. *Let X be a quasi-compact scheme with the property that every finite subset is contained in an open affine set. (Quasiprojective schemes over affine schemes have this property, for example.) Then there is a natural injective homomorphism $\operatorname{Br}(X) \hookrightarrow H^2(X_{et}, \mathbb{G}_m)$.*

The group $H^2(X_{et}, \mathbb{G}_m)$ is called the cohomological Brauer group, and often denoted $\operatorname{Br}'(X)$.

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Proof. This is Theorem IV.2.5 in Milne's Étale Cohomology [4]. The proof is very similar in spirit (if not in technical details) to the topological result in the last talk, so it will only be sketched.

The first step is to show that the sequence

$$0 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 0$$

is an exact sequence in the étale topology. This relies on a version of the Noether-Skolem theorem. Then one considers the connecting homomorphism in the long exact sequence of étale cohomology

$$\delta : H^1(X_{et}, \mathrm{PGL}_n) \to H^2(X_{et}, \mathbb{G}_m)$$

As in the topological case, this is non-Abelian étale cohomology, defined using cocycles. The technical conditions on X are to ensure that the derived functor cohomology matches the étale Čech cohomology. There are alternatives discussed in Milne using a more general theory.

Next, one identifies rank n^2 Azumaya algebras with $H^1(X_{et}, \mathrm{PGL}_n)$. Since the automorphism sheaf of Mat_n is PGL_n by the generalization of Noether-Skolem and Azumaya algebras are étale locally Mat_n, one can obtain a cocycle from an Azumaya algebra (details are found in the section of twisted forms in III.4). Furthermore, Azumaya algebras which are endomorphism algebras of locally free modules of rank n are the image of the map $H^1(X_{et}, \mathrm{GL}_n) \to H^2(X_{et}, \mathbb{G}_m)$ just as in the topological case.

Therefore on can map Azumaya algebras to $H^2(X_{et}, \mathbb{G}_m)$ by combining these maps for all n. A calculation with cocycles, similar to the one in the topological case, shows that this map turns the tensor product of Azumaya algebras into the product in $H^2(X_{et}, \mathbb{G}_m)$. The trivial Azumaya algebras are the image of the $H^1(X_{et}, \mathrm{GL}_n)$, so this factors to give an injective map from the Brauer group $\mathrm{Br}(X)$ to $H^2(X_{et}, \mathbb{G}_m)$. □

The same argument as in the topological case shows the Brauer group is torsion (at least provided X has finitely many connected components). There exist singular schemes where $H^2(X_{et}, \mathbb{G}_m)$ is not torsion, so the Brauer group is not identified with the cohomological Brauer group in all cases. According to Milne, there are no known examples where $\mathrm{Br}(X)$ is not $\mathrm{Br}'(X)[tors]$. In many familiar cases, they are provably the same, such as for smooth schemes and for local rings of dimension at most one [4, IV.2.15,17]. We will prove the case of Henselian local rings in the next section.

Example 5. We already have one case where we know the cohomological Brauer group equals the Brauer group: that of $\mathrm{Spec}\, k$. By analyzing central simple algebras over a field, we determined that $\mathrm{Br}(k) = H^2(G_k, (k^{\mathrm{sep}})^\times)$. On the other hand, $\mathrm{Br}(\mathrm{Spec}\, k)$ injects into $H^2((\mathrm{Spec}\, k)_{et}, \mathbb{G}_m) = H^2(G_k, (k^{\mathrm{sep}})^\times)$ by Grothendieck's Galois theory.

2. The Brauer Group of Henselian Rings

We aim to understand the Brauer group of Henselian local rings. The main result will be following: our proof is adapted from Milne [4, IV.2].

Theorem 6. *Let R be a Henselian ring. Then* $\mathrm{Br}(\mathrm{Spec}\, R) = \mathrm{Br}'(\mathrm{Spec}\, R)$.

The key is understanding when the injection $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ discussed previously is a surjection.

Proposition 7. *Let A be a local ring, $X = \mathrm{Spec}\, A$ and γ an element of the cohomological Brauer group $\mathrm{Br}'(X)$. Then γ lies in the image of $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ if and only if there is a finite étale surjective map $Y \to X$ such that γ maps to 0 in $\mathrm{Br}'(Y)$.*

Proof. We may pass to a further étale extension, so $Y = \mathrm{Spec}\, B$ may be assumed to be a Galois cover of X.[1] Recall the Hochsilde-Serre spectral sequence for this cover says there is a spectral sequence with $E_2^{p,q} = H^q(G, H^p(Y_{et}, \mathbb{G}_m))$ that converges to $H^n(X_{et}, \mathbb{G}_m)$. This spectral sequence follows from Grothendieck's composition of functors spectral sequence plus realizing that $H^p(G, \Gamma(Y, I)) =$

[1]This means the automorphisms of Y over X act simply transitively on fibers.

0 for any injective sheaf because this can be interpreted as Čech cohomology of I for the cover Y over X. Now as B is a Galois cover of local ring A, it is semilocal and hence we know that $H^1(Y_{et}, \mathbb{G}_m) = 0$ by interpretting it as the Picard group. Therefore $E_2^{1,q} = 0$ for all q, and the E_2 page looks like

$$\begin{array}{ccc} H^2(G, H^0(Y_{et}, \mathbb{G}_m)) & 0 & \dots \\ \\ H^1(G, H^0(Y_{et}, \mathbb{G}_m)) & 0 & H^1(G, H^2(Y_{et}, \mathbb{G}_m)) \\ \\ H^0(G, H^0(Y_{et}, \mathbb{G}_m)) & 0 & H^0(G, H^2(Y_{et}, \mathbb{G}_m)) \end{array}$$

We have that $E_\infty^{1,1} = 0$, $E_2^{0,2} = E_\infty^{0,2}$ and $E_3^{2,0} = E_2^{2,0}$ because of the vanishing column. But then $E_\infty^{2,0} = \ker(E_2^{2,0} \to E_3^{0,3})$, so $E_\infty^{2,0}$ is a submodule of $E_2^{2,0}$. The converges of the spectral sequence, plus the vanishing of $E_\infty^{1,1}$, gives the short exact sequence

$$0 \to H^2(G, H^0(Y_{et}, \mathbb{G}_m)) \to H^2(X_{et}, \mathbb{G}_m) \to E_\infty^{2,0} \to 0.$$

The hypothesis that an element $\gamma \in \mathrm{Br}'(X) = H^2(X_{et}, \mathbb{G}_m)$ is 0 after the extension to Y means that it maps to 0 in $H^0(G, H^2(Y_{et}, \mathbb{G}_m))$, so such elements are automatically coming from an element $\gamma' \in H^2(G, H^0(Y_{et}, \mathbb{G}_m))$. Conversely, every such element maps to 0 in $E_\infty^{2,0}$ and hence in $H^0(G, H^2(Y_{et}, \mathbb{G}_m))$. But an element $H^2(G, H^0(Y_{et}, \mathbb{G}_m))$ can be represented by a two cocycle with coefficients in $B^\times$. Just as in the case of central simple algebras, we can use this cocycle to write down the multiplication law on an Azumaya algebra over X that splits over Y, and conversely. □

We also need a lemma about étale cohomology [4, III.3.11a].

Lemma 8. *Let G be a smooth quasi-projective group scheme over a Helselian ring A. Let $X = \operatorname{Spec} A$ and G_0 be the base change of G over the inclusion of the closed point X_0 into X. Then $H^i(X_{et}, G) = H^i((X_0)_{et}, G_0)$ for $i \geq 1$.*

We can now prove Theorem 6. Let R be a Henselian ring, k its residue field, and K its field of fractions. Let γ be an element of $\mathrm{Br}'(\operatorname{Spec} R)$, $\mathfrak{m}$ the closed point of $\operatorname{Spec} R$, with residue field k. The lemma tells us that $H^2((\operatorname{Spec} R)_{et}, \mathbb{G}_m) = H^2((\operatorname{Spec}(k))_{et}, \mathbb{G}_m)$ via the natural map. Therefore γ corresponds to an element of $\mathrm{Br}(k)$, which we have identified with Galois cohomology and étale cohomology. But by the theory of central simple algebras, this is split over some finite separable extension l/k. This means it maps to 0 in $\mathrm{Br}(l)$. Pick a primitive element for l/k, and lift its minimal polynomial to R. Adjoin a root to K, creating a field extension L/K, and let S be the integral closure of R in L. Since R is Henselian, any finite R-algebra is a product of local rings. Therefore S is a local ring, with residue field l. Then the commuting diagram

$$\begin{array}{ccc} H^2((\operatorname{Spec} S)_{et}, \mathbb{G}_m) & \xrightarrow{\sim} & \mathrm{Br}(l) \\ \uparrow & & \uparrow \\ H^2((\operatorname{Spec} R)_{et}, \mathbb{G}_m) & \xrightarrow{\sim} & \mathrm{Br}(k) \end{array}$$

shows that γ maps to 0 in $H^2((\operatorname{Spec} S)_{et}, \mathbb{G}_m) = \mathrm{Br}'(\operatorname{Spec} S)$. □

Theorem 1 is immediate: both R and its quotient are Henselian, and

$$\mathrm{Br}'(\operatorname{Spec} R) = H^2((\operatorname{Spec} R)_{et}, \mathbb{G}_m) = H^2((\operatorname{Spec} k)_{et}, \mathbb{G}_m) = \mathrm{Br}'(\operatorname{Spec} k)$$

3. The Auslander-Brumer Theorem

This is of a commutative algebra flavor, and uses Galois cohomology instead of the Azumaya algebras used by Grothendieck. We first need a compatibility theorem between the Brauer group of a discrete valuation ring and the cohomological definition that Auslander and Brumer study. Throughtout, R is a discrete valuation ring, K is its field of fractions, L is the maximal unramified extension of K, S the integral closure of R in L, and $G = \mathrm{Gal}(L/K)$.

Proposition 9. *With the previous notation,* $\mathrm{Br}(\mathrm{Spec}\, R) = H^2(G, S^\times)$.

Proof. This uses an explicit understanding of Galois cohomology over a DVR and its relation to étale cohomology. This is presented in detail in Stein's notes on Galois cohomology [5], which follow Mazur's discussion of Galois cohomology of number fields. In particular, if , one shows that $H^2((\mathrm{Spec}\, R)_{et}, \mathbb{G}_m) = H^2(G, S^\times)$ by combining Theorem 27.6 with the interpretation of the étale sheaf $\mathbb{G}_m$ as a Galois module over the DVR in example 23.5. Thus $\mathrm{Br}'(\mathrm{Spec}\, R) = H^2(G, S^\times)$. The cohomological Brauer group equals the Brauer group in this situation: we proved this only for Henselian rings, but it holds more generally for local rings of dimension 1 [4, IV.2.17]. □

Furthermore, we need a finer result on the splitting of central simple algebras.

Proposition 10. *Every central simple algebra over a non-archimedean local field K is split by an unramified extension.*

Proof. This is a relatively standard step in the approach to local class field theory via Brauer groups - additional details are found in Milne [3, IV.4]. The idea is to mimick basic algebraic number theory in the ring of integers of a division algebra. Let D be a division algebra over K. Let $\mathcal{O}_K$ be the ring of integers and k the residue field of K. From the theory of central simple algebras, we know that $[D : K] = n^2$. One checks that there is an extension of the valuation to R, and then defines the ring of integers $\mathcal{O}_D$ and its maximal ideal $\mathfrak{m}_D$ as usual. Furthermore, we know that $l = \mathcal{O}_D/\mathfrak{m}_D$ is a finite division algebra and hence a field. Picking a primitive element and lifting to $\alpha \in D$, $L = K(\alpha)$ is a subfield of D. Thus $f = [l : k] = [L : K] \leq n$, since the maximal subfield in D is of dimension n. Since $K(\alpha)$ is unramified over K, we just need to show that $K(\alpha)$ splits D. By the theory of central simple algebras, this happens if L is maximal, ie if $f = n$.

We can also look at the ramification of this division algebra: the usual proofs go through and show that $\mathfrak{m}_D = (\mathfrak{m}_K \mathcal{O}_D)^e$ for some integer e. The exponent e is the ramification degree. By filtering $\mathcal{O}_D \supset \mathfrak{m}_D \supset \mathfrak{m}_D^2 \supset \ldots \supset \mathfrak{m}_K \mathcal{O}_D$, we see that each quotient has dimension f over k, while the chain is of length e. Thus $\mathcal{O}_D/\mathfrak{m}_K\mathcal{O}_D$ has dimension ef. Since D is of dimension n^2 over K, we see that $ef = n^2$. Thus $e = f = n$. □

Now we construct a short exact sequence of Galois cohomology.

Proposition 11. *There is a split short exact sequence*

$$0 \to H^2(G, \mathcal{O}_L^\times) \to H^2(G, L^\times) \to H^2(G, \mathbb{Z}) \to 0.$$

Proof. For any finite unramified extension M of K, we have a short exact sequence of $G_M = \mathrm{Gal}(M/K)$ modules

$$0 \to \mathcal{O}_M^\times \to M^\times \to \mathbb{Z} \to 0.$$

It is split by choosing a uniformizer $\pi \in M^\times$. Taking the long exact sequence in group cohomology, we see that

$$0 \to H^2(G_M, \mathcal{O}_M^\times) \to H^2(G_M, M^\times) \to H^2(G_M, \mathbb{Z}) \to 0$$

is exact: the other degrees do not contribute as the short exact sequence was split. Now take the limit over all unramified extensions. □

We can now prove the Auslander-Brumer theorem by identifying the cohomology groups in this sequence. Since $H^2(G, \mathbb{Z}) \simeq H^1(G, \mathbb{Q}/\mathbb{Z})$ (consider the connecting homomorphism in $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$), the rightmost term is naturally $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$. Since the extension is the maximal unramified one, G is the Galois group of the residue extension, G_k. Thus $H^2(G, \mathbb{Z}) \simeq \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) = X(\mathbb{G}_k)$. Proposition 9 shows $H^2(G, \mathcal{O}_L^\times) = \mathrm{Br}(R)$. Finally, recall we showed that $H^2(G_K, (K^{\mathrm{sep}})^\times)$ is the Brauer group of a field by taking the limit over finite separable extensions of central simple algebras split by the extension, which were described cohomologically. Since every central simple algebra split over some separable extension, the limit is the Brauer group. Since we know that in this case every central simple algebra splits over an unramified extension, we may use the maximal unramified extension instead and still obtain the Brauer group. This completes the proof. □

References

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