EVALUATIONS OF CUBIC TWISTED KLOOSTERMAN SHEAF SUMS

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Abstract. We prove some conjectures of R. Evans and N. Katz presented in a paper by Evans regarding twisted Kloosterman sheaf sums T_n. These conjectures give explicit evaluations of the sums T_n where the character is cubic and $n = 4$. There are also conjectured relationships between evaluations of T_n and the coefficients of certain modular forms. For three of these modular forms, each of weight 3, it is conjectured that the coefficients are related to the squares of the coefficients of weight 2 modular forms. We prove these relationships using the theory of complex multiplication.

1. Introduction and Statement of Results

In a recent paper [2], Evans proves results relating hypergeometric functions over finite fields to twisted sums T_n related to Kloosterman sheaves [5]. Evans was motivated by empirical observations of Katz on explicit evaluations of T_n in terms of coefficients of modular forms.

To make this precise, let p be prime, q be a power of p, and k and ℓ be integers. For a multiplicative character $C : \mathbb{F}_q^\times \to \mathbb{C}^\times$ and for $a \in \mathbb{F}_q^\times$, define a *twisted Kloosterman sum* to be

$$K(C^k, a) := \sum_{x \in \mathbb{F}_q^\times} C^k(x) \zeta_p^{x + a\overline{x}}, \tag{1.1}$$

where $\overline{x}$ denotes the multiplicative inverse of x in $\mathbb{F}_q^\times$ and $\zeta_p^u := \exp\left(\frac{2\pi i \mathrm{Trace}(u)}{p}\right)$. Further, define the corresponding *twisted Kloosterman sheaf sum* to be

$$T_n(C, k, \ell) := \sum_{a \in \mathbb{F}_q^\times} \overline{C}^\ell(a)(\pi_1^n + \pi_1^{n-1}\pi_2 + \cdots + \pi_1\pi_2^{n-1} + \pi_2^n), \tag{1.2}$$

where $\overline{C} := C^{-1}$. Here $\pi_1 := \pi_1(a, k)$ and $\pi_2 := \pi_2(a, k)$ are defined to be the complex numbers (unique up to order) for which

$$K(C^k, a) = -\pi_1 - \pi_2 \text{ and } q \cdot C^k(-a) = \pi_1\pi_2. \tag{1.3}$$

This paper deals with the case $p = q$ as we study some conjectures of Evans/Katz [2, 3].

Conjecture (Evans/Katz). *Let $p \equiv 1 \pmod 3$ and let C be any cubic character on $\mathbb{F}_p$.*

1. *We have $T_4(C, 2, 1) = -2p^2$.*
2. *Let $r_3 := r_3(p)$ denote the unique integer that satisfies $r_3 \equiv 1 \pmod 3$ and, for some integer t_3, satisfies*
$$4p = r_3^2 + 27t_3^2.$$
If $J(\overline{C}, \overline{C})$ is the Jacobi sum for $\overline{C}$, and $G(C)$ the Gauss sum for C, then we have
$$T_4(C, 2, 0) = -r_3 J(\overline{C}, \overline{C}) G(C)^2.$$

We prove the following theorem

Theorem 1. *Evans's/Katz's conjecture is true.*

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In the same paper [2], there are further conjectured identities between coefficients of pairs of modular forms of weights 2 and 3. Let $q := e^{2\pi i z}$, where $\mathrm{Im}(z) > 0$. Denote the complex vector space of cusp forms of weight k, level N, and nebentypus χ with respect to $\Gamma_0(N)$ by $S_k(\Gamma_0(N), \chi)$. See, for example, [4] or [6] for introductory material to modular forms.

Conjecture (Evans/Katz)**.** *The following are true:*

(1) *For a quartic character χ_1 with conductor 35, let*

$$f_1(z) = \sum_{n=1}^{\infty} a_1(n)q^n \in S_3\left(\Gamma_0(35), \left(\frac{\cdot}{35}\right)\right) \text{ and}$$

$$g_1(z) = \sum_{n=1}^{\infty} c_1(n)q^n \in S_2\left(\Gamma_0(175), \chi_1\right)$$

be the newforms whose Fourier expansions begin with

$$f_1(z) = q + q^3 + 4q^4 - 5q^5 + 7q^7 - 8q^9 - 13q^{11} + \cdots$$
$$g_1(z) = q + \sqrt{-7}iq^3 + 2iq^4 + \sqrt{7}iq^7 - 4iq^9 - 3q^{11} - \cdots .$$

Then if p is a prime for which $\left(\frac{-35}{p}\right) = 1$, we have

$$a_1(p) = |c_1(p)|^2 - 2p. \tag{1.4}$$

Furthermore, if $p \equiv 1 \pmod 3$ is prime, then we have $a_1(p) = r_3 - T_7(C, 2, 1)/p^3$, where C is a cubic character.

(2) *For a quartic character χ_2 with conductor 1155, let*

$$f_2(z) = \sum_{n=1}^{\infty} a_2(n)q^n \in S_3\left(\Gamma_0(1155), \left(\frac{\cdot}{1155}\right)\right) \text{ and}$$

$$g_2(z) = \sum_{n=1}^{\infty} c_2(n)q^n \in S_2\left(\Gamma_0(5775), \chi_2\right)$$

be the newforms whose Fourier expansions begin with

$$f_2(z) = q + 3q^3 + 4q^4 + 5q^5 - 7q^7 + 9q^9 + 11q^{11} + \cdots$$
$$g_2(z) = q + \sqrt{-3}iq^3 + 2iq^4 - \sqrt{7}iq^7 - 3iq^9 - \sqrt{11}q^{11} + \cdots .$$

Then if $\left(\frac{33}{p}\right) = \left(\frac{-35}{p}\right) = 1$, we have

$$a_2(p) = |c_2(p)|^2 - 2p. \tag{1.5}$$

Furthermore, if $p \equiv 1 \pmod 3$, then $a_2(p) = r_3^3/p - 2r_3 - T_{11}(C, 1, 1)/p^5$, where C is any cubic character.

(3) *For a quartic character χ_3 with conductor 3003, let*

$$f_3(z) = \sum_{n=1}^{\infty} a_3(n)q^n \in S_3\left(\Gamma_0(3003), \left(\frac{\cdot}{3003}\right)\right) \text{ and}$$

$$g_3(z) = \sum_{n=1}^{\infty} c_3(n)q^n \in S_2\left(\Gamma_0(39039), \chi_3\right)$$

be the newforms whose Fourier expansion begin with

$$f_3(z) = q + 3q^3 + 4q^4 - 7q^7 + 9q^9 - 11q^{11} \cdots$$
$$g_3(z) = q + \sqrt{3}q^3 + 2iq^4 + \sqrt{-7}iq^7 + 3q^9 - \sqrt{-11}iq^{11} + \cdots .$$

Then if $\left(\frac{p}{21}\right) = \left(\frac{p}{143}\right) = \left(\frac{p}{5}\right)$, *we have*

$$\left(\frac{p}{13}\right) a_3(p) = |c_3(p)|^2 - 2p. \tag{1.6}$$

Furthermore, if $p \equiv 1 \pmod 3$, *we have*

$$a_3(p) = \left(\frac{p}{55}\right)\left(r_3^5/p^2 - 4r_3^3/p + 3r_3 - T_{15}(C,1,0)/p^7\right),$$

where C *is a cubic character on* $\mathbb{F}_p$.

Remark. *We provide a description of these modular forms in terms of Hecke characters: the above q-expansions were calculated using this description. They are in the same space of modular forms, but not always equal, to the forms in the original conjecture. For example, our* f_3 *is a* $13-$*quadratic twist of the form in the original conjecture.*

We prove the following theorem. It implies (1.4) - (1.6) since in the course of the proof it will become clear that $c_i(p)^2\overline{\chi_i}(p)$ is positive.

Theorem 2. *Assuming the notation above, the following are true:*

(1) *If* p *is a prime for which* $\left(\frac{-35}{p}\right) = 1$, *then*

$$a_1(p) = c_1(p)^2\overline{\chi_1}(p) - 2p$$

and if $\left(\frac{-35}{p}\right) = -1$, *then* $a_1(p) = c_1(p) = 0$.

(2) *If* $\left(\frac{-1155}{p}\right) = 1$, *then*

$$a_2(p) = \left(\frac{p}{35}\right)\left(c_2(p)^2\overline{\chi_2}(p) - 2p\right)$$

and if $\left(\frac{-1155}{p}\right) = -1$, *then* $a_2(p) = c_2(p) = 0$.

(3) *If* $\left(\frac{-3003}{p}\right) = 1$, *then*

$$a_3(p) = \left(\frac{p}{13}\right)\left(c_3(p)^2\overline{\chi_3}(p) - 2p\right)$$

and if $\left(\frac{-3003}{p}\right) = -1$, *then* $a_3(p) = c_3(p) = 0$.

In Section 2, we give preliminaries about Gauss, Jacobi, and Kloosterman sums and then use these to prove Theorem 1. Section 3 introduces modular forms with complex multiplication and uses the fact that the forms in the conjecture have complex multiplication to prove Theorem 2.

2. Preliminaries on Gauss, Jacobi, and Kloosterman Sums

Note that if C is any character on $\mathbb{F}_q^\times$, we extend C to $\mathbb{F}_q$ by defining $C(0) = 0$. Further, if C is a cubic character on $\mathbb{F}_q$, then C and $C^2 = \overline{C}$ are the only cubic characters on $\mathbb{F}_q$. Throughout the rest of the paper, unless other conditions are noted, sums will run over $\mathbb{F}_p^\times$. We first recall some general facts about Gauss and Jacobi sums.

2.1. Gauss and Jacobi Sums.

We will be making use of elementary properties of Gauss and Jacobi sums. Given a character χ on $\mathbb{F}_q$, define the Gauss sum by

$$G(\chi) = \sum_{x\in\mathbb{F}_q} \chi(x)\zeta_p^x. \tag{2.1}$$

Now, let A and B be characters on $\mathbb{F}_q$ and define the Jacobi sum by

$$J(A,B) = \sum_{x\in\mathbb{F}_q} A(x)\, B(1-x). \tag{2.2}$$

Let ε denote the trivial character and ϕ denote the unique quadratic character on $\mathbb{F}_q$. Some classical properties of Gauss and Jacobi sums, stated in [1] as Theorem 1.1.4a and Theorem 2.1.3a, are

$$G(A)\, G(\overline{A}) = qA(-1) \tag{2.3}$$

$$J(A,B) = \frac{G(A)\, G(B)}{G(AB)}, \tag{2.4}$$

provided $A \neq \varepsilon$ in (2.3) and $AB \neq \varepsilon$ in (2.4). If $AB = \varepsilon$, then by Theorem 2.1.1c of [1]

$$J(A,\overline{A}) = -A(-1) \tag{2.5}$$

Furthermore, Theorem 2.1.4 of [1] states that if $A \neq \varepsilon$ then

$$A(4)\, J(A,A) = J(A,\phi). \tag{2.6}$$

From Table 3.1.1 of [1], we have the following useful lemma specific to cubic characters.

Lemma 3. *If $p \equiv 1 \pmod 3$ is prime and C is a cubic character on $\mathbb{F}_p^\times$, then*

$$J(C,C) + J(\overline{C},\overline{C}) = r_3.$$

2.2. **Preliminaries on Moments of Kloosterman Sums.** We now prove several lemmas vital to the proof of Theorem 1.

Lemma 4. *If $p \equiv 1 \pmod 3$ is prime and C is a cubic character on $\mathbb{F}_p^\times$, then*

$$\sum_{a\in\mathbb{F}_p^\times} C(a) K(C^2,a)^2 = p^2 - 2p.$$

Proof. Consider the substitution $y \mapsto xy$, $a \mapsto x^2 a$, so

$$\sum_a C(a)K(C^2,a)^2 = \sum_a C(a) \sum_{x,y} C^2(xy)\zeta_p^{x+y+a(\overline{x}+\overline{y})} =$$

$$\sum_a C(a) \sum_{x,y} C^2(x^3 y)\zeta_p^{x(1+y+a(1+\overline{y}))} = \sum_{a,y} C(ay^2) \sum_x \zeta_p^{x(1+y+a(1+\overline{y}))}.$$

When $1+y+a(1+\overline{y}) \neq 0$, as x runs through $\mathbb{F}_p^\times$ we have $\zeta_p^{x(1+y+a(1+\overline{y}))}$ runs through the p^{th} roots of unity except for 1. Thus the above expression equals

$$-\sum_{1+y+a(1+\overline{y})\neq 0} C(ay^2) + (p-1) \sum_{1+y+a(1+\overline{y})=0} C(ay^2) = -\sum_{a,y} C(ay^2) + p \sum_{1+y+a(1+\overline{y})=0} C(ay^2)$$

$$= p \sum_{1+y+a(1+\overline{y})=0} C(ay^2).$$

When $1+y+a(1+\overline{y}) = 0$, if $y=-1$, then a can take on any nonzero value, so

$$p \sum_{\substack{y=-1\\ a}} C(ay^2) = p \sum_a C(a) = 0.$$

Otherwise, $a = -\dfrac{1+y}{1+\overline{y}}$, which gives

$$p \sum_{1+y+a(1+\overline{y})=0} C(ay^2) = p \sum_{y\neq -1} C\left(-y^2 \frac{1+y}{1+\overline{y}}\right) = p \sum_{y\neq -1} C\left(y^3 \frac{1+y}{1+y}\right) = p\,(p-2).$$ ☐

Lemma 5. *For C as in Lemma 4, we have*

$$\sum_{a\in\mathbb{F}_p^\times}\overline{C}(a)K(C^2,a)^4=2p^3-7p^2.$$

Proof. Write

$$\sum_a\overline{C}(a)K(C^2,a)^4=\sum_a C^2(a)\sum_{x,y,z,w}C^2(xyzw)\zeta_p^{x+y+z+w+a(\overline{x}+\overline{y}+\overline{z}+\overline{w})}.$$

Using the change of variables $a\mapsto w^2a$, $x\mapsto wx$, $y\mapsto wy$, $z\mapsto wz$, this becomes

$$\sum_{a,x,y,z}\sum_w C^2(w^6axyz)\zeta_p^{w(1+x+y+z+a(1+\overline{x}+\overline{y}+\overline{z}))}.$$

For ease of notation, define $R(a,x,y,z):=1+x+y+z+a(1+\overline{x}+\overline{y}+\overline{z})$. Then the sum splits into

$$\begin{aligned}&\sum_{R(a,x,y,z)\neq 0}C^2(axyz)\sum_{w\in\mathbb{F}_p^\times}\zeta_p^{wR(a,x,y,z)}+\sum_{R(a,x,y,z)=0}C^2(axyz)\sum_w\zeta_p^{w\cdot 0}\\&=-\sum_{R(a,x,y,z)\neq 0}C^2(axyz)+(p-1)\sum_{R(a,x,y,z)=0}C^2(axyz)\\&=-\sum_{a,x,y,z}C^2(axyz)+p\sum_{R(a,x,y,z)=0}C^2(axyz)=p\sum_{R(a,x,y,z)=0}C^2(axyz),\end{aligned}$$

as $\sum_{a,x,y,z}C^2(axyz)=0$. The condition $R(a,x,y,z)=0$ (i.e. $1+x+y+z=-a(1+\overline{x}+\overline{y}+\overline{z})$) implies $a=-\frac{1+x+y+z}{1+\overline{x}+\overline{y}+\overline{z}}$ unless $1+\overline{x}+\overline{y}+\overline{z}=0$, in which case $1+x+y+z=0$ and a may be an arbitrary nonzero element. Since $C^2(-1)=1$ and $C^2(0)=0$, the sum becomes

$$\begin{aligned}&=p\sum_{1+\overline{x}+\overline{y}+\overline{z}\neq 0}C^2\left(\frac{1+x+y+z}{1+\overline{x}+\overline{y}+\overline{z}}\,xyz\right)+p\sum_{1+\overline{x}+\overline{y}+\overline{z}=0}C^2(xyz)\sum_a C^2(a)\\&=p\sum_{1+\overline{x}+\overline{y}+\overline{z}\neq 0}C^2\left(\frac{1+x+y+z}{1+\overline{x}+\overline{y}+\overline{z}}\,xyz\right).\end{aligned}$$

It suffices to show that $\sum_{1+\overline{x}+\overline{y}+\overline{z}\neq 0}C^2\left(\frac{1+x+y+z}{1+\overline{x}+\overline{y}+\overline{z}}xyz\right)=2p^2-7p$. Making the change of variable $y\mapsto yz$, the sum under consideration becomes

$$\sum_{\substack{(x,y)\neq(-1,-1)\\ z\neq-\frac{\overline{y}+1}{\overline{x}+1}}}C^2\left(\frac{x+1+z(y+1)}{\overline{x}+1+\overline{z}(\overline{y}+1)}xyz^2\right)=\sum_{(x,y)\neq(-1,-1)}C^2(xy)\sum_{z\neq-\frac{\overline{y}+1}{\overline{x}+1}}C^2\left(\frac{x+1+z(y+1)}{\overline{y}+1+z(\overline{x}+1)}\right).$$

Before evaluating this sum, first examine the inner sum. Notice that $\frac{x+1+z(y+1)}{\overline{y}+1+z(\overline{x}+1)}$ is a fractional linear transformation, so it is invertible if the determinant of the corresponding matrix is nonzero. The determinant is zero precisely when

$$(y+1)(\overline{y}+1)-(x+1)(\overline{x}+1)=0.$$

That is, when $y=x$ or $y=\overline{x}$. When this expression is invertible, the quotient takes on all values of $\mathbb{F}_p$ except for the values corresponding to $z=0$ and $z=\infty$, namely $\frac{x+1}{\overline{y}+1}$ and $\frac{y+1}{\overline{x}+1}$. Thus, for a fixed x and y, the inner sum becomes

$$\sum_{z\neq-\frac{\overline{y}+1}{\overline{x}+1}}C^2\left(\frac{x+1+z(y+1)}{\overline{y}+1+z(\overline{x}+1)}\right)=-\left(C^2\left(\frac{x+1}{\overline{y}+1}\right)+C^2\left(\frac{y+1}{\overline{x}+1}\right)\right). \tag{2.7}$$

First we handle the exceptional cases. In the case $x = \overline{y} \neq -1$, we have

$$\sum_{z \neq -\frac{\overline{y}+1}{\overline{x}+1}} C^2\left(\frac{x+1+z(y+1)}{\overline{y}+1+z(\overline{x}+1)}\right) = \sum_{z \neq -\frac{x+1}{\overline{x}+1}} C^2\left(\frac{\overline{y}+1+z(y+1)}{\overline{y}+1+z(y+1)}\right) = p-2.$$

If $x = y \neq -1$, it follows that

$$\sum_{z \neq -\frac{\overline{y}+1}{\overline{x}+1}} C^2\left(\frac{y+1+z(y+1)}{\overline{y}+1+z(\overline{y}+1)}\right) = \sum_{z \neq -\frac{\overline{x}+1}{\overline{x}+1}} C^2(y) \cdot C^2\left(\frac{y+1+z(y+1)}{y+1+z(y+1)}\right) = C^2(y) \sum_{z \neq -1} 1 = C^2(y)(p-2).$$

Combining these two special cases, we have

$$\begin{aligned} S \;:=& \sum_{\substack{x=y \text{ or } x=\overline{y} \\ x \neq -1}} C^2(xy) \sum_{z \neq -\frac{\overline{y}+1}{\overline{x}+1}} C^2\left(\frac{x+1+z(y+1)}{\overline{y}+1+z(\overline{x}+1)}\right) \\ =& \sum_{\substack{x=y \\ x \neq -1}} C^2(y^3)(p-2) + \sum_{\substack{x=\overline{y} \\ x \neq \pm 1}} C^2(xy)(p-2) = (p-2)(2p-5), \end{aligned}$$

where $x \neq 1$ in the second summation to avoid double counting the case $x = y = \overline{y}$. Define T by

$$T \;:= \sum_{\substack{x=y \text{ or } x=\overline{y} \\ x \neq -1}} C^2(xy) C^2\left(\frac{y+1}{\overline{x}+1}\right). \tag{2.8}$$

When $x = y$ and $x \neq 0, -1$, the summand in (2.8) simplifies as follows:

$$C^2(xy) C^2\left(\frac{y+1}{\overline{x}+1}\right) = C^2(x^2) C^2\left(\frac{x+1}{\overline{x}+1}\right) = C^2(x^3) = 1.$$

When $x = \overline{y}$ and $x \neq 0, -1$, the summand is

$$C^2(xy) C^2\left(\frac{y+1}{\overline{x}+1}\right) = C^2(1) C^2\left(\frac{\overline{x}+1}{\overline{x}+1}\right) = 1.$$

Thus we have

$$T = \sum_{\substack{x=y \\ x \neq -1}} 1 + \sum_{\substack{x=\overline{y} \\ x \neq \pm 1}} 1 = (p-2) + (p-3) = 2p-5.$$

Finally, we can evaluate

$$\begin{aligned} -2 \sum_{x,y \neq -1} C^2(xy) C^2\left(\frac{y+1}{\overline{x}+1}\right) &= -2 \sum_{y} C^2(y(y+1)) \sum_{x} C(x(x+1)) \\ &= -2 \sum_{y \in \mathbb{F}_p} C^2(-y) C^2(1-y) \sum_{x \in \mathbb{F}_p} C(-x) C(1-x) \\ &= -2\, J(C^2, C^2)\, J(C, C) = -2p. \end{aligned} \tag{2.9}$$

In the last step we employ equations (2.3) and (2.4). Then, using equation (2.7) and the two special cases S and T, we have

$$\sum_{(x,y)\neq(-1,-1)} C^2(xy) \sum_{z\neq -\frac{\overline{y}+1}{\overline{x}+1}} C^2\left(\frac{x+1+z(y+1)}{\overline{y}+1+z(\overline{x}+1)}\right) = -\sum_{\substack{x\neq y \text{ and } x\neq\overline{y}\\(x,y)\neq(-1,-1)}} C^2(xy)\left(C^2\left(\frac{y+1}{\overline{x}+1}\right)+C^2\left(\frac{x+1}{\overline{y}+1}\right)\right)+S$$

$$= -2\sum_{\substack{x\neq y \text{ and } x\neq\overline{y}\\(x,y)\neq(-1,-1)}} C^2(xy)C^2\left(\frac{y+1}{\overline{x}+1}\right)+S = -2\sum_{x,y\neq -1} C^2\left(xy\,\frac{y+1}{\overline{x}+1}\right)+2T+S$$

$$= -2\sum_{x,y\neq -1} C^2\left(xy\frac{y+1}{\overline{x}+1}\right)+p(2p-5) = 2p^2-7p.$$

The last step uses (2.9). This establishes the lemma. ∎

Lemma 6. *If $p\equiv 1 \pmod 3$ is prime and C is a cubic character on $\mathbb{F}_p^\times$, then*

$$\sum_{a\in\mathbb{F}_p^\times} C^2(a)K(C^2,a)^2 = -G(C)^2.$$

Proof. By definition (1.1), we have

$$\sum_a C^2(a)K(C^2,a)^2 = \sum_a C^2(a)\sum_{x,y} C^2(xy)\zeta_p^{x+y+a(\overline{x}+\overline{y})}$$

$$= \sum_{x,y} C^2(xy)\zeta_p^{x+y}\sum_a C^2(a)\zeta_p^{a(\overline{x}+\overline{y})} = \sum_{\overline{x}+\overline{y}\neq 0} C^2\left(\frac{xy}{\overline{x}+\overline{y}}\right)\zeta_p^{x+y}G(C^2)+\sum_{x=-y} C^2(xy)\sum_a C^2(a)$$

$$= G(C^2)\sum_{y\neq -x} C^2\left(\frac{xy}{\overline{x}+\overline{y}}\right)\zeta_p^{x+y}.$$

Making the change of variables $y\mapsto xy$, when $y\neq -1$, $\zeta_p^{x(1+y)}$ takes on all values except for 1 as x varies. Hence by (2.4) we have

$$G(C^2)\sum_{y\neq -1}\sum_x C^2\left(\frac{y}{1+\overline{y}}\right)\zeta_p^{x(1+y)} = -G(C^2)\sum_{y\neq -1} C^2\left(\frac{y}{1+\overline{y}}\right)$$

$$= -G(C^2)\sum_{y\neq 1} C(y)C(1-y) = -G(C^2)\,J(C,C) = -G(C)^2$$

∎

Lemma 7. *If $p\equiv 1 \pmod 3$ is prime and C is a cubic character on $\mathbb{F}_p^\times$, then*

$$\sum_{a\in\mathbb{F}_p^\times} K(C^2,a)^4 = -p\,G(C^2)\left(4J(C,C)+J(\overline{C},\overline{C})\right).$$

Proof. We write

$$\sum_a K(C^2,a)^4 = \sum_{a,x,y,z,w} C^2(xyzw)\zeta_p^{x+y+z+w+a(\overline{x}+\overline{y}+\overline{z}+\overline{w})} = \sum_{x,y,z,w} C^2(xyzw)\zeta_p^{x+y+z+w}\sum_a \zeta_p^{a(\overline{x}+\overline{y}+\overline{z}+\overline{w})}.$$

Notice that when $\overline{x}+\overline{y}+\overline{z}+\overline{w}\neq 0$, the inner sum is -1, and when $\overline{x}+\overline{y}+\overline{z}+\overline{w}=0$ the inner sum is $p-1$. So we can rewrite our sum as

$$\begin{aligned} &-\sum_{\overline{x}+\overline{y}+\overline{z}+\overline{w}\neq 0} C^2(xyzw)\zeta_p^{x+y+z+w} + (p-1)\sum_{\overline{x}+\overline{y}+\overline{z}+\overline{w}=0} C^2(xyzw)\zeta_p^{x+y+z+w} \\ &= p\sum_{\overline{x}+\overline{y}+\overline{z}+\overline{w}=0} C^2(xyzw)\zeta_p^{x+y+z+w} - \sum_{x,y,z,w} C^2(xyzw)\zeta_p^{x+y+z+w}. \end{aligned} \tag{2.10}$$

Using the definitions and properties of Gauss and Jacobi sums in equations (2.3) and (2.4), we have that the second sum satisfies

$$\sum_{x,y,z,w} C^2(xyzw)\zeta_p^{x+y+z+w} = G(C^2)^4 = p\,G(C^2)\,J(\overline{C},\overline{C}). \tag{2.11}$$

It suffices to show that

$$\sum_{\overline{x}+\overline{y}+\overline{z}+\overline{w}=0} C^2(xyzw)\zeta_p^{x+y+z+w} = -4G(C^2)\,J(C,C). \tag{2.12}$$

Making the change of variables $y\mapsto xy$, $z\mapsto xz$, and $w\mapsto xw$ in equation (2.12), we find

$$\begin{aligned} \sum_{\overline{x}+\overline{y}+\overline{z}+\overline{w}=0} C^2(xyzw)\zeta_p^{x+y+z+w} &= \sum_{1+\overline{y}+\overline{z}+\overline{w}=0}\sum_x C^2(xyzw)\zeta_p^{x(1+y+z+w)} \\ &= \sum_{\substack{1+\overline{y}+\overline{z}+\overline{w}=0\\ 1+y+z+w\neq 0}} C^2\left(\frac{yzw}{1+y+z+w}\right)G(C^2) + \sum_{\substack{1+\overline{y}+\overline{z}+\overline{w}=0\\ 1+y+z+w=0}}\sum_x C^2(xyzw). \end{aligned}$$

The second term is zero, so we consider the first term. Solving $1+\overline{y}+\overline{z}+\overline{w}=0$ for w and substituting, this becomes

$$\begin{aligned} \sum_{\substack{1+\overline{y}+\overline{z}+\overline{w}=0\\ 1+y+z+w\neq 0}} C^2\left(\frac{yzw}{1+y+z+w}\right)G(C^2) &= G(C^2)\sum_{\substack{1+y+z-\frac{yz}{yz+z+y}\neq 0\\ yz+y+z\neq 0}} C^2\left(\frac{(yz)^2}{yz+z+y}\right)C^2\left(\frac{1}{1+y+z-\frac{yz}{yz+z+y}}\right) \\ &= G(C^2)\sum_{yz+z+y\neq 0} C\left(yz(yz+z+y)\right)C\left(1+y+z-\frac{yz}{yz+y+z}\right). \end{aligned}$$

Making the change of variables $y+z\mapsto a$, $zy\mapsto b$, this becomes

$$G(C^2)\left(\sum_{a,b} C(b)C(a+ab+a^2) + \sum_{a,b} C(b)C(a+ab+a^2)\phi(a^2-4b) - \sum_{a+b=0} C(a)C(a)(1+\phi(a^2+4a))\right). \tag{2.13}$$

The third sum is

$$\sum_a C(a^2) + \sum_a C(a^2)\phi(a^2-4a) = \sum_a C(1/a)\phi(1-4/a) = C(1/4)\sum_a C(a)\phi(1-a).$$

Using equation (2.6) we find that the third sum is the following Jacobi sum

$$\sum_{a+b=0} C(a)C(a)\left(1+\phi(a^2+4a)\right) = J(C,C). \tag{2.14}$$

The first sum in equation (2.13), after making the substitution $b\mapsto -b(a+1)$, gives

$$\sum_{a,b} C(-b)C(1-b)C(a)C(1+a)^2 = J(C,C)J(C,C^2) = -J(C,C) \tag{2.15}$$

using (2.5).

To evaluate the second sum in equation (2.13), we will first show that given a fixed $b \in \mathbb{F}_p^\times$

$$\sum_{a\in\mathbb{F}_p} C(1+a+a^2b) = \phi(1-4b)C\left(\frac{1-4b}{4b}\right)J(C,\phi). \tag{2.16}$$

Making the change of variable $1+a+a^2b \mapsto x$, if $1-4b \neq 0$ the sum evaluates as

$$\begin{aligned}\sum_{x\in\mathbb{F}_p} C(x)\left(1+\phi(1-4b+4bx)\right) &= \phi(1-4b)\sum_{x\in\mathbb{F}_p} C(x)\phi\left(1+\frac{4b}{1-4b}x\right)\\ &= \phi(1-4b)\sum_{x\in\mathbb{F}_p} C\left(\frac{1-4b}{4b}x\right)\phi(1-x) = \phi(1-4b)C\left(\frac{1-4b}{4b}\right)J(C,\phi).\end{aligned}$$

Note that if $1-4b=0$, then both sides of equation (2.16) are zero. Next, making the substitution $b\mapsto a^2b$ in the second sum of equation (2.13) and using equation (2.16) gives

$$\begin{aligned}&\sum_b C(b)\phi(1-4b)\sum_a C(1+a+a^2b) = \sum_b C(b)\phi(1-4b)\left(\phi(1-4b)C\left(\frac{1-4b}{4b}\right)J(C,\phi)-1\right)\\ &= \sum_b C\left(\frac{1-4b}{4}\right)J(C,\phi) - \sum_b C(b)\phi(1-4b) = -C(1/4)\,J(C,\phi) - C(1/4)\,J(C,\phi) = -2\,J(C,C).\end{aligned}$$

Combining this with equations (2.14) and (2.15), it follows from (2.13) that

$$\sum_{\overline{x}+\overline{y}+\overline{z}+\overline{w}=0} C^2(xyzw)\zeta_p^{x+y+z+w} = -4\,G(C^2)\,J(C,C). \qquad \square$$

2.3. **Proof of Theorem 1.** We now use the lemmas from Section 2 to prove Theorem 1.

Proof. Recall that for fixed k and a, $\pi_1\cdot\pi_2 = p\,C^k(-a)$ and $\pi_1+\pi_2 = -K(C^k,a)$. When $k=2$ and C is a cubic character, an elementary manipulation shows

$$\pi_1^4+\pi_1^3\pi_2+\pi_1^2\pi_2^2+\pi_1\pi_2^3+\pi_2^4 = K(C^2,a)^4 - 3p\,C^2(a)K(C^2,a)^2 + C(a)p^2.$$

Then by Lemmas 4 and 5 we have

$$\begin{aligned}T_4(C,2,1) &= \sum_a \overline{C}(a)K(C^2,a)^4 - 3p\sum_a C(a)K(C^2,a)^2 + \sum_a p^2\\ &= 2p^3-7p^2-3p(p(p-2))+(p-1)p^2 = -2p^2.\end{aligned}$$

This proves the first statement of Theorem 1.

For the second half of the theorem, we employ Lemmas 6 and 7 to obtain

$$\begin{aligned}T_4(C,2,0) &= \sum_a K(C^2,a)^4 - 3p\sum_a C^2(a)K(C^2,a)^2 + p^2\sum_a C(a)\\ &= p\,G(C^2)(-4\,J(C,C)-J(\overline{C},\overline{C})) + 3p\,G(C)^2\\ &= p\,G(C^2)(-J(C,C)-J(\overline{C},\overline{C})),\end{aligned}$$

where the last step uses equation (2.4). Using Lemma 3, equations (2.3) and (2.4) yield

$$T_4(C,2,0) = p\,G(C^2)(-r_3) = -G(C^2)^2G(C)r_3 = -G(C)^2J(C^2,C^2)r_3. \qquad \square$$

3. Proof of Theorem 2

3.1. **Modular Forms with Complex Multiplication.** Here we recall some basic facts about modular forms with complex multiplication (see [6] for more on modular forms). Let $K = \mathbb{Q}\left(\sqrt{-D}\right)$ be an imaginary quadratic field with discriminant $-D$. Let $\mathcal{O}_K$ be its ring of integers, $\mathfrak{m}$ a nontrivial ideal and $I_\mathfrak{m}$ be the group of fractional ideals relatively prime to $\mathfrak{m}$. A *Hecke character* for K is a group homomorphism

$$\phi : I_\mathfrak{m} \to \mathbb{C}^\times$$

such that for all $\alpha \in K^\times$ with $\alpha \equiv 1 \pmod{\mathfrak{m}}$, ϕ satisfies

$$\phi(\alpha\mathcal{O}_K) = \alpha^{k-1},$$

for some $k \in \mathbb{Z}$ with $k \geq 2$. Define a Dirichlet character χ_ϕ for n relatively prime to $\mathfrak{m}$ by

$$\chi_\phi(n) := \phi((n))/n^{k-1}$$

which has modulus $N\mathfrak{m}$. Given a Hecke character, ϕ, we obtain a modular form with complex multiplication. More precisely, if we let

$$\Phi(z) := \sum_{\mathfrak{a}} \phi(\mathfrak{a}) q^{N(\mathfrak{a})} = \sum_{n=1}^{\infty} a(n) q^n, \tag{3.1}$$

where $N(\mathfrak{a})$ denotes the norm of the ideal $\mathfrak{a}$ and the first sum is over integral ideals $\mathfrak{a} \subset \mathcal{O}_K$ that are prime to $\mathfrak{m}$, then $\Phi(z)$ is a cusp form in $S_k(\Gamma_0(|D| \cdot N(\mathfrak{m})), \left(\frac{-D}{\cdot}\right)\chi_\phi)$.

Here is a standard method to construct Hecke characters for an imaginary quadratic field $\mathbb{Q}\left(\sqrt{-D}\right)$ in which ± 1 are the only units. Let $\pi_1, \ldots, \pi_m$ be a minimal set of ideals whose ideal classes generate the class group. Let the order of π_i in the class group be n_i. Given $\mathfrak{m}$, we must have $\phi((\alpha)) = \alpha^{k-1}$ if $\alpha \equiv 1 \pmod{\mathfrak{m}}$. Let χ be a character on $\mathcal{O}_K/\mathfrak{m}$ extended to $\mathcal{O}_K$ that satisfies $\chi(-1) = (-1)^{k-1}$. First define ϕ on principal ideals by $\phi((\alpha)) = \chi(\alpha)\alpha^{k-1}$. Since the only units in $\mathcal{O}_K$ are ± 1, this definition is independent of the choice of generator, α. To extend ϕ to non-principal ideals, and thus obtain a Hecke character, it suffices to define it on $\pi_1, \ldots, \pi_m$ and extend multiplicatively. By the above assumptions, π_i is non-principal and $\pi_i^{n_i} = (\alpha)$ for some $\alpha \in K^\times$. Thus $\phi(\pi_i)$ must be one of the n_i^{th} roots of $\phi((\alpha)) = \alpha^{k-1}\chi(\alpha)$. Fixing $\phi(\pi_i)$ for each i gives the Hecke character. Having fixed the $\phi(\pi_i)$ subject to the above constraint, extending multiplicatively yields

$$\phi(\pi) = \frac{\alpha^{k-1}\chi(\alpha)}{\phi(\pi_1)^{s_1} \cdots \phi(\pi_m)^{s_m}} \tag{3.2}$$

where $\pi\, \pi_1^{s_1} \, \ldots \, \pi_m^{s_m} = (\alpha)$ and $0 \leq s_i < n_i$.

3.2. **Proof of the Conjecture.** We now prove Theorem 2 by showing that the specified modular forms have complex multiplication. The relationship between coefficients follows from this description.

Proof. We prove the theorem in three parts.

Part (1). Let $K = \mathbb{Q}\left(\sqrt{-35}\right)$, which has ring of integers $\mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{-35}}{2}\right]$, discriminant -35, and class number 2. A prime $(p) \subset \mathbb{Z}$ splits in $\mathcal{O}_K$ if $\left(\frac{-35}{p}\right) = 1$, is ramified if $p = 5, 7$, and is inert otherwise. Let π_3 be the prime ideal $\left(3, \frac{1+\sqrt{-35}}{2}\right)$ lying above 3. Then π_3 generates the class group.

The form $f_1(z)$ has complex multiplication coming from a Hecke character ϕ_1 with $k = 3$ and $\mathfrak{m} = \mathcal{O}_K$. The character χ is trivial, and there are two Hecke characters depending on how ϕ_1 is defined on the non-principal ideal π_3. Setting $\phi_1(\pi_3) = \frac{1+\sqrt{-35}}{2}$ gives the Hecke character whose associated modular form conjecturally agrees with the values of the twisted Kloosterman sheaf sum. By the discussion above, this defines a modular form f_1 of weight 3 and level $35 \cdot N\mathfrak{m} = 35$. The Hecke character gives a Dirichlet character $\chi_{\phi_1}(n) = \phi_1((n))/n^2 = 1$, so the nebentypus of the modular form is $\left(\frac{-35}{\cdot}\right)$. If (p) is inert, there are no ideals of norm p so $a_1(p) = 0$. If $\left(\frac{-35}{p}\right) = 1$, then (p) splits; so there are two ideals of norm p in $\mathcal{O}_K$, namely the prime ideals such that $(p) = \pi\overline{\pi}$. Thus the coefficient of q^p is $a_1(p) = \phi_1(\pi) + \phi_1(\overline{\pi})$.

Similarly, $c_1(p)$ comes from a Hecke character ϕ_2 with $k = 2$ and $\mathfrak{m} = \pi_5$, where $\pi_5^2 = (5)$. Let χ be a quartic character defined on $\mathcal{O}_K/\pi_5$, and extend it to a character on $\mathcal{O}_K$. This induces a Hecke character ϕ_2 once we extend to non-principal ideals. It gives a Dirichlet character χ which is a quartic character of conductor 5. Thus ϕ_2 gives a weight 2 modular form using equation (3.1). Since $N\mathfrak{m} = 5$, it is a cusp form with weight 2, level 175, and nebentypus $\chi_1 := \left(\frac{-35}{\cdot}\right)\chi$ (a quartic character with conductor 35). If (p) is inert, then $c_1(p) = 0$, whereas if $(p) = \pi\overline{\pi}$, the coefficient of q^p is $c_1(p) = \phi_2(\pi) + \phi_2(\overline{\pi})$. Picking χ so that $\chi(2) = i$ gives the coefficients listed in the conjecture.

It remains to check that $a_1(p) = c_1(p)^2\overline{\chi}(p) - 2p$ when $(p) = \pi\overline{\pi}$. We will first show that if $p = \alpha\overline{\alpha}$ with $\alpha \in \mathcal{O}_K$ then

$$\frac{\phi_1((\alpha))}{\phi_2((\alpha))^2} = \chi(p), \qquad \frac{\phi_1(\pi_3)}{\phi_2(\pi_3)^2} = -\chi(3).$$

The second is a direct calculation. To see the first, note that

$$\frac{\phi_1((\alpha))}{\phi_2((\alpha))^2} = \frac{\alpha^2}{\chi(\alpha)^2\alpha^2} = \frac{1}{\chi(\alpha)^2} = \chi(\alpha\overline{\alpha}) = \chi(p)$$

where the last steps use the fact that $\chi(\alpha^2) = \pm 1$ since χ is quartic and the fact that $\chi(\alpha) = \chi(\overline{\alpha})$ since every equivalence class modulo π_5 has a rational integer representative and $\mathfrak{m} = \pi_5 = \overline{\pi_5}$.

Now let $\pi\overline{\pi} = (p)$ and $\pi = (\alpha)\pi_3^{s_3}$. This gives us the following equation:

$$\frac{\phi_1(\pi)}{\phi_2(\pi)^2} = \chi(p)(-1)^{s_3}.$$

The character $\overline{\chi_1}$ appearing in Theorem 2 part (1) agrees with $\overline{\chi}$ when evaluated at p since $\left(\frac{-35}{p}\right) = 1$. Then we know that

$$\begin{aligned} c_1(p)^2\overline{\chi}(p) - 2p &= \phi_2(\pi)^2\overline{\chi}(p) + \phi_2(\overline{\pi})^2\overline{\chi}(p) + 2\phi_2((p))\overline{\chi}(p) - 2p \\ &= (-1)^{s_3}\overline{\chi}^2(p)(\phi_1(\pi)^2 + \phi_1(\overline{\pi})^2) + 2p\chi(p)\overline{\chi}(p) - 2p = (-1)^{s_3}\left(\frac{p}{5}\right)a_1(p). \end{aligned}$$

since χ^2 is the quadratic character on the residue field which has 5 elements. However, if (p) splits, it splits into non-principal ideals only if $\left(\frac{p}{5}\right) = -1$. To see this, consider a non-principal ideal π lying above p. Then $\pi = (\alpha)\pi_3$ and $3N\alpha = p$. Since $N\alpha = \pm 1, 0 \mod 5$ and (p) is prime and unramified, $\left(\frac{N\alpha}{5}\right) = 1$ and hence $\left(\frac{p}{5}\right) = \left(\frac{3}{5}\right) = -1$. Thus we can conclude

$$c_1(p)^2\overline{\chi}(p) - 2p = (-1)^{s_3}\left(\frac{p}{5}\right)a_1(p) = a_1(p).$$

Part (2). We work in the field $\mathbb{Q}\left(\sqrt{-1155}\right)$, which has class group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Generators are given by $\pi_3 = \left(3, \frac{3+\sqrt{-1155}}{2}\right)$, $\pi_{17} = \left(17, -1 + \sqrt{-1155}\right)$, and $\pi_{19} = \left(19, 2 + \sqrt{-1155}\right)$. It turns out that $f_2(z)$ has complex multiplication where $\mathfrak{m} = \mathcal{O}_K$. To get agreement with the conjecture relating this form to Kloosterman sheaf sums, let $\phi_1(\pi_3) = 3$, $\phi_1(\pi_{17}) = \frac{-1+\sqrt{-1155}}{2}$, and $\phi_1(\pi_{19}) = \frac{17-\sqrt{-1155}}{2}$.

The Hecke character ϕ_2 is defined with $\mathfrak{m} = \pi_5$ and χ a quartic character on the residue field with $\chi(2) = i$. As in the first part, we get modular forms of weights 3 and 2 with coefficients $a_2(p) = \phi_1(\pi) + \phi_1(\overline{\pi})$ and $c_2(p) = \phi_2(\pi) + \phi_2(\overline{\pi})$ when $(p) = \pi\overline{\pi}$.

Doing a similar calculation as in part (1) we find that if $\alpha\overline{\alpha} = p$ then

$$\frac{\phi_1((\alpha))}{\phi_2((\alpha))^2} = \chi(p), \qquad \frac{\phi_1(\pi_3)}{\phi_2(\pi_3)^2} = -\chi(3).$$
$$\frac{\phi_1(\pi_{17})}{\phi_2(\pi_{17})^2} = -\chi(17), \qquad \frac{\phi_1(\pi_{19})}{\phi_2(\pi_{19})^2} = -\chi(19).$$

If $\pi\overline{\pi} = (p)$ and

$$\pi = (\alpha)\pi_3^{s_3}\pi_{17}^{s_{17}}\pi_{19}^{s_{19}}, \text{ then} \tag{3.3}$$
$$\frac{\phi_1(\pi)}{\phi_2(\pi)^2} = \chi(p)(-1)^{s_3+s_{17}+s_{19}}.$$

Then we can conclude that

$$\begin{aligned} c_2(p)^2\,\overline{\chi}(p) - 2p &= \phi_2(\pi)^2\overline{\chi}(p) + \phi_2(\overline{\pi})^2\overline{\chi}(p) + 2\phi_2((p))\overline{\chi}(p) - 2p \\ &= (-1)^{s_3+s_{17}+s_{19}}\overline{\chi}(p)^2(\phi_1(\pi) + \phi_1(\overline{\pi})). \end{aligned}$$

In this case, $\overline{\chi}^2(p) = \chi^2(p) = \left(\frac{p}{5}\right)$, so taking norms of (3.3) shows that $\left(\frac{p}{7}\right) = (-1)^{s_3+s_{17}+s_{19}}$. Thus when (p) splits $a_2(p) = \left(\frac{p}{35}\right)\left(c_2(p)^2\overline{\chi}(p) - 2p\right)$.

Part (3). For the third statement, we work in the field $\mathbb{Q}\left(\sqrt{-3003}\right)$. The class group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, with generators $\pi_3 = \left(3, \frac{3-\sqrt{-3003}}{2}\right)$, $\pi_{29} = \left(29, 10 + \sqrt{-3003}\right)$, and $\pi_{31} = \left(31, 2 + \sqrt{-3003}\right)$. Define the Hecke character ϕ_1 to give a weight 3 form using $\mathfrak{m} = (1)$. To make it agree with the conjectured relationship with Kloosterman sheaf sums, set $\phi_1(\pi_3) = 3$, $\phi_1(\pi_{29}) = \frac{19-\sqrt{-3003}}{2}$, and $\phi_1(\pi_{31}) = \frac{29-\sqrt{-3003}}{2}$. To define ϕ_2, let $\mathfrak{m}$ be the prime above 13, let χ be a quartic character on the residue field with $\chi(2) = i$, and proceed as before. The same sort of calculations show that if $\alpha\overline{\alpha} = p$ then

$$\frac{\phi_1((\alpha))}{\phi_2((\alpha))^2} = \chi(p), \qquad \frac{\phi_1(\pi_3)}{\phi_2(\pi_3)^2} = \chi(3),$$
$$\frac{\phi_1(\pi_{29})}{\phi_2(\pi_{29})^2} = \chi(29), \qquad \frac{\phi_1(\pi_{31})}{\phi_2(\pi_{31})^2} = \chi(31).$$

Thus when $(p) = \pi\overline{\pi}$ we have

$$\begin{aligned} c_3(p)^2\,\overline{\chi}(p) - 2p &= \phi_2(\pi)^2\overline{\chi}(p) + \phi_2(\overline{\pi})^2\overline{\chi}(p) + 2\phi_2(p)\overline{\chi}(p) - 2p \\ &= \overline{\chi}^2(p)(\phi_1(\pi) + \phi_1(\overline{\pi})) = \left(\frac{p}{13}\right)a_3(p). \end{aligned}$$

□

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References

[1] B. Berndt, R. Evans, and K. Williams, *Gauss and Jacobi sums*, Canadian Math. Soc. Monographs **21**, Wiley Interscience, New York, 1998.

[2] R.J. Evans, *Hypergeometric* ${}_3F_2(1/4)$ *evaluations over finite fields and Hecke eigenforms*, Proc. Amer. Math. Soc. to appear.

[3] R.J. Evans, email communication, June, 2009.

[4] H. Iwaniec, Topics in Classical Automorphic Forms, Amer. Math. Soc., Providence, RI, 1997.

[5] N. Katz, *Gauss sums, Kloosterman sums, and monodromy groups*, Annals of Mathematics Studies 116, Princeton University Press, Princeton, NJ, 1988.

[6] K. Ono, *The web of modularity: Arithmetic of the coefficients of modular forms and q-series*, CBMS No.102, Amer. Math. Soc., Providence, RI, 2004.

[7] SAGE Mathematical Software, http://www.sagemath.org

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