Geometric Deformations of Orthogonal and Symplectic Galois Representations

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Chapter 1

Introduction

Galois representations are central in modern number theory, perhaps most famously in the proof of Fermat's Last Theorem. There are natural sources of Galois representations in algebraic geometry, and the Langlands program conjecturally connects them with automorphic forms. For example, given a classical modular eigenform there is an associated two-dimensional Galois representation which we say is *modular*. Another example comes from an elliptic curve defined over $\mathbf{Q}$: the absolute Galois group of the rationals acts on the p-adic Tate module, giving a two-dimensional Galois representation over the p-adic integers. Fermat's Last Theorem follows from the fact that the Galois representations arising from elliptic curves are modular.

More generally, Galois representations arise from algebraic geometry through the action of the Galois group on the p-adic étale cohomology of a variety defined over a number field. Such representations are ramified at finitely many primes and are potentially semistable above p. An absolutely irreducible p-adic Galois representation satisfying these latter properties is said to be *geometric*. The *Fontaine-Mazur conjecture* asserts that all geometric Galois representations arise as subsets of Tate twists of p-adic étale cohomology. A more refined version has been proven in the two-dimensional case: two-dimensional geometric Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ are modular.

A key technique in answering these questions is the deformation theory of a residual Galois representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_n(\mathbf{F}_q)$. Understanding the universal deformation ring (the ring $R^{\square}_{\overline{\rho}}$ which parametrizes lifts of $\overline{\rho}$ to complete Noetherian local rings with residue field $\mathbf{F}_q$) gives results about Galois representations. Residual representations are also interesting in their own right. In the two-dimensional case, *Serre's conjecture* states that every odd irreducible representation is the reduction of a modular Galois representation. This was proven by Khare and Winterberger. Generalizations of Serre's conjecture to groups other than GL_2 have been proposed, most recently by Gee, Herzig, and Savitt [GHS15]. The combination of Serre's conjecture and the Fontaine-Mazur conjecture motivates the study of when a residual representation admits a geometric lift to characteristic zero.

In particular, we are interested in the following situation. Let K be a finite extension of $\mathbf{Q}$ with absolute Galois group Γ_K. Suppose k is a finite field of characteristic p, $\mathcal{O}$ the ring of integers in a p-adic field with residue field $\mathcal{O}/\mathfrak{m} = k$, and G is a connected reductive group defined over $\mathcal{O}$. For a continuous representation $\overline{\rho} : \Gamma_K \to G(k)$, in light of these conjectures it is important to study when there exists a continuous representation $\rho : \Gamma_K \to G(\mathcal{O})$ lifting $\overline{\rho}$ that is geometric in a suitable sense over $\mathcal{O}[\frac{1}{p}]$ (using $G \hookrightarrow \mathrm{GL}_n$).

Suppose $K = \mathbf{Q}$. When $G = \mathrm{GL}_2$, Ramakrishna developed a technique to produce geometric lifts [Ram02]. His results provided evidence for Serre's conjecture. In the symplectic and orthogonal cases, we generalize this method and prove the existence of geometric lifts in favorable conditions. Two essential hypotheses are that $\overline{\rho}$ is odd (as defined in §2.5.1) and that $\overline{\rho}$ restricted to the decomposition group at p looks like the reduction of a crystalline representation with *distinct* Hodge-Tate weights. For precise statements, see Theorem 1.1.2.2 and Theorem 2.5.3.4. This provides evidence for generalizations of Serre's conjecture. In contrast, when $G = \mathrm{GL}_n$ with $n > 2$, the representation $\overline{\rho}$ cannot be odd, and the method does not apply. In such cases, there is no expectation that such lifts exist.

Ramakrishna's method works by establishing a local-to-global result for lifting Galois representations subject to local constraints. Let ρ be a lift of $\overline{\rho}$ to $\mathcal{O}/\mathfrak{m}^n$. Provided a cohomological obstruction vanishes, it is possible to lift ρ over $\mathcal{O}/\mathfrak{m}^{n+1}$ subject to local constraints if (and only if) it possible to lift $\rho|_{\Gamma_v}$ over $\mathcal{O}/\mathfrak{m}^{n+1}$ for all v in a fixed set of places of $\mathbf{Q}$ containing p and the places where $\overline{\rho}$ is ramified. In Chapter 2, we generalize Ramakrishna's argument so it can be applied to any connected reductive group and show that the cohomological obstruction vanishes if we allow controlled ramification at additional places at which $\overline{\rho}$ is unramified.

It remains to pick local deformation conditions at p and at places where $\overline{\rho}$ is ramified which are liftable in the sense that it is always possible to suitably lift $\rho|_{\Gamma_v}$. At p, we define a *Fontaine-Laffaille deformation condition* in Chapter 3 by using deformations arising from Fontaine-Laffaille modules that carry extra data corresponding to a symmetric or alternating pairing.

At a prime $\ell \neq p$ where $\overline{\rho}$ is ramified, we generalize the minimally ramified deformation condition defined for GL_n in [CHT08, §2.4.4]. In simple cases, this deformation condition controls the ramification of ρ by controlling deformations of a unipotent element $\overline{u}$ of $\mathrm{GL}_n(k)$. There is a natural parabolic k-subgroup containing $\overline{u}$, and the deformation condition is analyzed by deforming this parabolic subgroup and then lifting $\overline{u}$ inside this subgroup. This idea does *not* work for other algebraic groups. In Example 1.2.3.2 and §4.4.3, we discuss an explicit example in GSp_4 where the analogous deformation based on parabolics is provably not liftable. In Chapter 4, we define a *minimally ramified deformation condition* by instead requiring that $\overline{u}$ deform so that "it lies in the same unipotent orbit as $\overline{u}$," and explain that this is a general phenomenon. The discovery and study of this deformation condition at ramified places $\ell \neq p$ (see §1.2.3) is the key innovation in this thesis. For GL_n, our notion agrees with minimally ramified deformation of [CHT08], but for other groups it is a genuinely different (liftable) deformation condition.

1.1 Motivation and Results

We will now review the background and state the results in more detail.

1.1.1 Background about Galois Representations

Let p be a prime.

Definition 1.1.1.1. A Galois representation $\rho : \Gamma_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is *modular* if there exists a modular eigenform f such that the Galois representation associated to f at the prime p is isomorphic to ρ. A residual Galois representation $\overline{\rho} : \Gamma_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is *modular* if it is the reduction of a modular representation ρ.

Serre's conjecture is one of the ingredients which motivated Ramakrishna's lifting result for two dimensional representations.

Fact 1.1.1.2 (Serre's Conjecture)**.** *Let $\overline{\rho} : \Gamma_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be an odd irreducible representation. Then $\overline{\rho}$ is modular.*

Remark 1.1.1.3. More precise versions give the weight and level of an associated modular form. This conjecture was proven by Khare and Winterberger [KW09a] [KW09b].

Let K be a number field with absolute Galois group Γ_K. Next we define two notions necessary to state the Fontaine-Mazur conjecture.

Definition 1.1.1.4. An irreducible representation of Γ_K over $\overline{\mathbf{Q}}_p$ *comes from geometry* if it is isomorphic to a sub-quotient of $H^i_{\text{ét}}(X_{\overline{K}}, \overline{\mathbf{Q}}_p(r))$ for some variety X over K and some $r \in \mathbf{Z}$.

Definition 1.1.1.5. A Galois representation $\rho : \Gamma_K \to G(\overline{\mathbf{Q}}_p)$ is *geometric* if it is unramified outside of a finite set of places and is potentially semi-stable (equivalently, de Rham) at all places of K above p.

Conjecture 1.1.1.6 (Fontaine-Mazur Conjecture)**.** *Let $\overline{\rho} : \Gamma_K \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be an irreducible representation. It is geometric if and only if it comes from geometry.*

When $n = 2$ and $K = \mathbf{Q}$, any irreducible geometric representation that is not the Tate-twist of an even representation factoring through a finite quotient of $\Gamma_{\mathbf{Q}}$ is modular up to a Tate twist.

The full Fontaine-Mazur conjecture is an open problem, but the statement in the case that $n = 2$ and $K = \mathbf{Q}$ follows from work of Kisin and Emerton [Kis09] [Eme11].

Motivated by these conjectures, Ramakrishna developed a lifting technique that produces geometric deformations.

Fact 1.1.1.7. *Under certain technical conditions, if $\overline{\rho} : \Gamma_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is an odd irreducible representation for which inertia at p acts as a fundamental character of level two, then there exists a geometric lift that is crystalline at p.*

This is the main theorem of [Ram02], which further develops the lifting technique of [Ram99]. Given this result, Serre's conjecture is a consequence of the two-dimensional version of the Fontaine-Mazur conjecture. Ramakrishna's result was seen as good evidence for Serre's conjecture before its proof by Khare and Winterberger.

1.1.2 Geometric Lifts

We are interested in generalizations of Ramakrishna's lifting result to reductive groups beyond GL_2, in particular symplectic and orthogonal groups. Generalizations of Serre's conjecture have been proposed in this setting, and most of the effort has been to find the correct generalization of the oddness condition and the weight (see for example the discussion in [GHS15], especially §2.1). The general flavor of these generalizations is that an odd irreducible Galois representation will be automorphic in the sense that it appears in the cohomology of an $\overline{\mathbf{F}}_p$-local system on a Shimura variety. For a general reductive group, there is no expectation that such representations will lift to characteristic zero. For example, as discussed in [CHT08, §1] the Taylor-Wiles method would work only if

$$[K : \mathbf{Q}] \, (\dim G - \dim B) = \sum_{v | \infty} \dim H^0(\mathrm{Gal}(\overline{K}_v / K_v), \mathrm{ad}^0(\overline{\rho})) \qquad (1.1.2.1)$$

where B is a Borel subgroup of G and $\mathrm{ad}^0(\overline{\rho})$ is the adjoint representation of Γ_K on the Lie algebra of the derived group of G. Only under such a "numerical coincidence" do we expect to obtain automorphy lifting theorems, and hence expect geometric lifts. This coincidence cannot hold for GL_n when $n > 2$, but can hold for $G = \mathrm{GSp}_{2n}$ and $G = \mathrm{GO}_m$, and for the group $\mathcal{G}_n$ related to GL_n considered in [CHT08].

We will define the notion of an *odd* Galois representations in §2.5.1, for which (1.1.2.1) holds. This will imply that K is totally real and for each archimedean place v of K and any complex conjugation c_v, the invariant subspace of the action of $\overline{\rho}(c_v)$ on $\mathrm{ad}^0(\overline{\rho})$ has "minimal" possible dimension (equal to $\dim G - \dim B$). There are odd representations for symplectic and orthogonal groups, but no odd representations for GL_n when $n > 2$. Thus we can hope to construct geometric lifts of odd representations valued in GO_m or GSp_m.

Remark 1.1.2.1. Our argument naturally uses *connected* reductive groups, so there are issues when GO_m is disconnected. In the orthogonal case we work with representations valued in GO_m° to avoid this issue, although as discussed in Remark 2.5.3.2 it is also possible to modify the argument to apply to some disconnected reductive groups.

Consider the reductive $\mathcal{O}$-group scheme $G = \mathrm{GO}_m^\circ$ or $G = \mathrm{GSp}_m$ and a (continuous) residual representation $\overline{\rho} : \Gamma_K \to G(k)$. In favorable circumstances, we will produce geometric lifts of $\overline{\rho}$. The exact hypotheses needed are somewhat complicated. We will state a simple version now, and defer a more detailed statement to Theorem 2.5.3.4.

Suppose that p is unramified in K and $p > 16m$. Let G' be the derived group of G, and assume that $G'(k) \subset \overline{\rho}(\Gamma_K)$. Furthermore suppose that $\overline{\rho}$ is odd as defined in §2.5.1. At places v above p, assume that $\overline{\rho}|_{\Gamma_{K_v}}$ is torsion crystalline with Hodge-Tate weights in an interval of length $\frac{p-2}{2}$, so it is Fontaine-Laffaille (these notions will be reviewed in §3.1). Furthermore, suppose that for each $\mathbf{Z}_p$-embedding of $\mathcal{O}_{K_v}$ in $\mathcal{O}_{\overline{K_v}}$, the Fontaine-Laffaille weights for $\overline{\rho}|_{\Gamma_{K_v}}$ with respect to that embedding are pairwise distinct.

Let $\mu : G \to \mathbf{G}_m$ be the similitude character, and define $\overline{\nu} = \mu \circ \overline{\rho} : \Gamma_K \to k^\times$. Suppose there is a lift $\nu : \Gamma_K \to W(k)^\times$ that is Fontaine-Laffaille at all places above p.

Theorem 1.1.2.2. *Under these assumptions, there exists a geometric lift $\rho : \Gamma_K \to G(\mathcal{O})$ of $\overline{\rho}$ where $\mathcal{O}$ is the ring of integers in a finite extension of $\mathbf{Q}_p$ with residue field containing k such that $\mu \circ \rho = \nu$. In* *particular, ρ is ramified at finitely many places of K, and for every place v of K above p the representation $\rho|_{\Gamma_{K_v}}$ is Fontaine-Laffaille and hence crystalline.*

1.1.3 Relation to Previous Work

Ramakrishna developed his lifting technique when $K = \mathbf{Q}$ and $G = \mathrm{GL}_2$ in [Ram99] and [Ram02], and produced geometric lifts. There have been various reformulations and generalizations of this that our results build on. In particular, the formalism developed in [Tay03] (still in the case of GL_2) suggested that it should be possible to generalize the technique to algebraic groups beyonds GL_2. Attempts were made in [Ham08] and [Man09] to generalize the result to GL_n, but ran into the obstruction that there were no odd representations for $n > 2$. The results in [Ham08] simply assume the existence of liftable local deformation conditions which probably do not exist, but do provide a nice model for the arguments in Chapter 2 generalizing Ramakrishna's method. In contrast, [Man09] constructs local deformation conditions but does not aim to produce geometric lifts.

For groups beyond GL_n, [CHT08] gave a lifting result for a group $\mathcal{G}_n$ related to GL_n but which admits odd representations. Studying the local deformation conditions for $\mathcal{G}_n$ reduced to studying representations valued in GL_n. At primes above p, [CHT08] studied a deformation condition based on Fontaine-Laffaille theory which is generalized in Chapter 3. The idea of doing so goes back to [Ram93]. (They also discussed a deformation condition based on the notion of ordinary representations which is not used in their lifting result). At primes not above p but where $\overline{\rho}$ is ramified, they defined a *minimally ramified* deformation condition, which we generalize in Chapter 4; this generalization is non-obvious and is our main innovation.

Building on this, Patrikis' unpublished undergraduate thesis [Pat06] explored Ramakrishna's method for symplectic groups. In particular, it generalized Ramakrishna's method to the group GSp_n, and generalized the Fontaine-Laffaille deformation condition at p. It did not generalize the minimally ramified deformation condition, so can only be applied to residual representations $\Gamma_{\mathbf{Q}} \to \mathrm{GSp}_n(k)$ which are unramified away from p, a stringent condition. Our results at p in Chapter 3 are a mild generalization of Patrikis' study of the Fontaine-Laffaille deformation condition.

More recently, Patrikis used Ramakrishna's method to produce geometric representations with exceptional monodromy [Pat15]. This involves generalizing Ramakrishna's method to any connected reductive group G and then modifying the technique to deform a representation valued in the principal $\mathrm{SL}_2 \subset G$ (coming from a modular form) to produce a geometric lift with Zariski-dense image. Chapter 2 is similar but independent of Patrikis' generalization of Ramakrishna's method. Our extensive study of local deformations conditions is not needed in [Pat15]: as the goal there is just to produce examples of geometric representations with exceptional monodromy, he could avoid generalizing the minimally ramified deformation condition.

Remark 1.1.3.1. There is also a completely different technique to produce lifts based on automorphy lifting theorems. For example, Khare and Winterberger use it in their proof of Serre's conjecture: see [KW09b, §4] especially the proof of Corollary 4.7. The finite generation needed in that argument comes from relating the Galois deformation ring to a Hecke algebra.

1.2 Overview of the Proof

There are three main steps in the proof Theorem 1.1.2.2. In Chapter 2 we generalize Ramakrishna's method to split connected reductive groups beyond GL_2; this reduces the problem to defining appropriate liftable local deformation conditions. The second step is carried out in Chapter 3, where at places above p we use Fontaine-Laffaille theory to produce a liftable deformation condition. Finally in Chapter 4 we reformulate and generalize a deformation condition at places above p studied in [CHT08, §2.4.4] for GL_n to obtain a liftable deformation condition for GSp_{2n} and GO_m.

1.2.1 Generalizing Ramakrishna's Method

Looking at the reformulation of Ramakrishna's method in [Tay03], it is not surprising that the method generalizes to algebraic groups beyond GL_2. It was generalized in [CHT08] to the group

$$\mathcal{G}_n = (\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \{1, j\} \quad \text{where} \quad j(g,\mu)j^{-1} = (\mu(g^{-1})^T, \mu).$$

We generalize the method to apply to any connected reductive group, although the method can only produce geometric lifts for groups for which there exist odd representations and for which we have nice local deformation conditions. Stefan Patrikis independently carried out this generalization [Pat15]. His results on the formalism are similar, with some minor variation in the technical hypotheses, but don't address local deformation conditions we require in Chapter 4.

Fix a prime p and finite field k of characteristic p. Let S be a finite set of places of a number field containing the places above p and the archimedean places, and define Γ_S to be the Galois group of the maximal extension of K unramified outside of S. Consider a continuous representation $\overline{\rho} : \Gamma_S \to G(k)$ where G is a split connected reductive group scheme over the ring of integers $\mathcal{O}$ in a p-adic field with maximal ideal $\mathfrak{m}$ whose residue field contains k. We assume that p is very good for G (Definition 2.1.1.1), so according to Lemma 2.1.1.6 the Lie algebra of the derived group of G is a direct summand of the Lie algebra of G: we denote this summand with adjoint action of Γ_K by $\mathrm{ad}^0(\overline{\rho})$. The cohomology of this Galois module controls the deformation theory of $\overline{\rho}$.

The hope would be to use deformation theory of produce $\rho_n : \Gamma_S \to G(\mathcal{O}/\mathfrak{m}^n)$ such that $\rho_1 = \overline{\rho}$, ρ_n lifts ρ_{n-1} for $n \geq 2$, and such that ρ_n satisfies a deformation condition at places above p for which the inverse limit

$$\rho = \varprojlim \rho_n : \Gamma_S \to G(\mathcal{O})$$

restricted to the decomposition group Γ_v would be a lattice in a de Rham (or crystalline) representation for places v of K above p. This inverse limit would then be the desired geometric lift of $\overline{\rho}$. Only after a careful choice of local deformation conditions and enlarging the set S will this work. Furthermore, defining these deformation conditions may require making an extension of k, which is harmless for our applications and is why we only require that the residue field of $\mathcal{O}$ merely contains k.

Proposition 2.5.2.1 shows a lifts exist subject to a global deformation condition $\mathcal{D}_S$ provided the dual Selmer group $H^1_{\mathcal{D}_S^\perp}(\Gamma_S, \mathrm{ad}^0(\overline{\rho})^*)$ vanishes. This Galois cohomology group is defined in (2.2.2.1), and encodes information about all of the local deformation conditions imposed. When it vanishes, there exists a lift of ρ_n to ρ_{n+1} satisfying local deformation conditions for $v \in S$ provided there exists lifts of $(\rho_n)|_{\Gamma_v}$ satisfying the deformation condition for all $v \in S$. This can be expressed as a local-to-global principle for lifting Galois representations with an obstruction lying in the cohomology group $H^1_{\mathcal{D}_S^\perp}(\Gamma_S, \mathrm{ad}^0(\overline{\rho})^*)$.

Corollary 2.4.2.6 gives a deformation condition at places where $\overline{\rho}$ is unramified such that allowing $\overline{\rho}$ to deform subject to this condition (i.e. enlarging S to contain such places and defining a new $\mathcal{D}_S$) forces $H^1_{\mathcal{D}_S^\perp}(\Gamma_S, \mathrm{ad}^0(\overline{\rho})^*)$ to be zero. We call this new deformation condition *Ramakrishna's deformation condition*, and study it in §2.4. The places of K at which we define this condition are found using the Chebotarev density theorem: each additional place where we allow ramification subject to Ramakrishna's deformation condition decreases the dimension of the dual Selmer group. For such places to exist, we need non-zero classes in certain cohomology groups, whose existence relies on the local deformation conditions satisfying the tangent space inequality (2.2.2.2)

$$\sum_{v \in S} \dim L_v \geq \sum_{v \in S} \dim H^0(\Gamma_v, \mathrm{ad}^0(\overline{\rho})),$$

where L_v is the tangent space of the local deformation condition at v. Furthermore, $\overline{\rho}$ needs to a "big" representation in the sense of Definition 2.3.1.1 in order to define Ramakrishna's deformation condition. Being a big representation is a more precise set of technical conditions that are implied for large enough p by the condition that $G'(k) \subset \overline{\rho}(\Gamma_K)$ appearing in Theorem 2.5.3.4. We discuss big representations in §2.3.

Remark 1.2.1.1. Patrikis uses a different set of technical conditions on $\overline{\rho}$. He does not require that $\mathrm{ad}^0(\overline{\rho})$ be an absolutely irreducible representation of Γ_K, but instead imposes several weaker conditions (conditions (1), (5), and (6) of [Pat15, §5]).

For the tangent space inequality to hold, it is crucial that $\overline{\rho}$ be an *odd* representation. The deformation conditions we will use at places v where $\overline{\rho}$ is ramified satisfy $\dim H^0(\Gamma_v, \mathrm{ad}^0(\overline{\rho})) = \dim L_v$. Using the Fontaine-Laffaille deformation condition at places above p, the tangent space inequality becomes

$$[K : \mathbf{Q}](\dim G - \dim B) \geq \sum_{v | \infty} h^0(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))$$

where B is a Borel subgroup of G; this can only be satisfied if K is totally real and $\overline{\rho}$ is odd. Thus in order to use the local-to-global principle, the residual representation must be odd and the deformation conditions we use at places above p and places where $\overline{\rho}$ is ramified must be liftable.

Remark 1.2.1.2. In general, references for results about reductive group schemes over rings are to [Con14], which gives a self-contained development using more recent methods, even though the results are usually also found in the earlier [SGA3].

1.2.2 Fontaine-Laffaille Deformation Condition

Let K be a finite extension of $\mathbf{Q}_p$, and $\mathcal{O}$ be the ring of integers of a p-adic field L with residue field k such that L splits K over $\mathbf{Q}_p$. (The latter is always possible after extending k.) Ramakrishna's method requires a deformation condition $\mathcal{D}_{\overline{\rho}}$ for the residual representation $\overline{\rho} : \Gamma_K \to G(k)$ such that:

- $\mathcal{D}_{\overline{\rho}}$ is liftable;
- $\mathcal{D}_{\overline{\rho}}$ is large enough, in the precise sense that its tangent space has dimension

$$[K : \mathbf{Q}_p](\dim G - \dim B) + \dim_k H^0(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$$

  where B is a Borel subgroup of G;
- $\mathcal{D}_{\overline{\rho}}(\mathcal{O})$ consists of certain lattices in crystalline representations.

We will construct such a condition using Fontaine-Laffaille theory.

Fontaine-Laffaille theory, introduced in [FL82], provides a way to describe torsion-crystalline representations in terms of semi-linear algebra when p is unramified in K. In particular, it provides an exact, fully faithful functor T_{cris} from the category of filtered Dieudonné modules to the category of $\mathcal{O}[\Gamma_K]$-modules with continuous action (Fact 3.1.1.14), and describes the image. In [CHT08, §2.4.1], it is used to define a deformation condition for n-dimensional representations, where the allowable deformations of $\overline{\rho}$ are exactly the deformations of the corresponding Fontaine-Laffaille module. This requires the technical assumption that the representation $\overline{\rho}$ is torsion-crystalline with Hodge-Tate weights in an interval of length $p-2$ and furthermore that the Fontaine-Laffaille weights of $\overline{\rho}$ under each embedding of K into L are distinct (see Remark 3.1.2.6).

We will adapt these ideas to symplectic and orthogonal groups under the stronger assumption that the Fontaine-Laffaille weights lie in an interval of length $\frac{p-2}{2}$. For symplectic groups and $K = \mathbf{Q}_p$, this was addressed in Patrikis's undergraduate thesis [Pat06]: Chapter 3 is a mild generalization. The key idea is to introduce a symmetric or alternating pairing into the semi-linear algebra data. To do so, it is necessary to use (at least implicitly via statements about duality) the fact that the functor T_{cris} is compatible with tensor products. This requires the stronger assumption that the Fontaine-Laffaille weights lie in an interval of length $\frac{p-2}{2}$, which guarantees that the Fontaine-Laffaille weights of the tensor product lie in an interval of length $p-2$. Furthermore, it is crucial to use the covariant version of the Fontaine-Laffaille functor used in [BK90] instead of the contravariant version studied in [FL82] in order for this compatibility with tensor products to hold. For more details, see §3.1.2. Given this, it is then reasonably straightforward to check that T_{cris} is compatible with duality and hence to translate the (perfect) alternating or symmetric pairing of Galois representations into a (perfect) symmetric or alternating pairing of Fontaine-Laffaille modules.

For a coefficient ring R, define $D^{\mathrm{FL}}_{\overline{\rho}}(R)$ to be all representations $\rho : \Gamma_K \to G(R)$ lifting $\overline{\rho}$ and lying in the essential image of T_{cris}. To study this Fontaine-Laffaille deformation condition, it suffices to study Fontaine-Laffaille modules. In particular, to show that the deformation condition is liftable (i.e. that it is always possible to lift a deformation satisfying the condition through a square-zero extension), it suffices to show that a Fontaine-Laffaille module with distinct Fontaine-Laffaille weights together with a perfect (symmetric or skew-symmetric) pairing can always be lifted through a square zero extension. This is a complicated but tractable problem in semi-linear algebra: Proposition 3.2.2.1 shows this is always possible. It is relatively simple to lift the underlying filtered module and the pairing, and requires more care to lift the semi-linear maps $\varphi^i_M : M^i \to M$. Likewise, to understand the tangent space of the deformation condition it suffices to study deformations of the Fontaine-Laffaille module corresponding to $\overline{\rho}$ to the dual numbers. Again, the most involved step is understanding possible lifts of the semi-linear maps after choosing a lift of the filtration and the pairing.

Remark 1.2.2.1. The proof that $D_{\overline{\rho}}^{\mathrm{FL}}$ is liftable and the computation of the dimension of its tangent space both use in an essential way the hypothesis that for each embedding of K into L the Fontaine-Laffaille weights are pairwise distinct.

Remark 1.2.2.2. An alternative deformation condition to use at primes above p is a deformation condition based on the concept of an ordinary representation. This is studied for any connected reductive group in [Pat15, §4.1]. It is suitable for use in Ramakrishna's method, and can give lifting results for a different class of torsion-crystalline representations.

1.2.3 Minimally Ramified Deformation Condition

Let $\ell \neq p$ be primes, L be a finite extension of $\mathbf{Q}_\ell$, and k a finite field of characteristic p. For a split connected reductive group G over the valuation ring $\mathcal{O}$ of a p-adic field K with residue field k, consider a residual representation $\overline{\rho} : \Gamma_L \to G(k)$. Ramakrishna's method requires a "nice" deformation condition for $\overline{\rho}$. If $\overline{\rho}$ were unramified, the unramified deformation condition would work. The interesting case is when $\overline{\rho}$ is ramified: we would like to define a deformation condition of lifts which are "ramified no worse that $\overline{\rho}$," so the resulting deformation condition is liftable despite the fact that the unrestricted deformation condition for $\overline{\rho}$ may not be liftable. To be precise, we require a deformation condition $\mathcal{D}_{\overline{\rho}}$ such that $\mathcal{D}_{\overline{\rho}}$ is liftable *and* whose tangent space has dimension (at least) $\dim_k H^0(\Gamma_L, \mathrm{ad}^0(\overline{\rho}))$.

In the case that $G = \mathrm{GL}_n$, the minimally ramified deformation condition defined in [CHT08, §2.4.4] works. We will generalize this to a *minimally ramified deformation condition* for symplectic and orthogonal groups when $p > n$. Attempting to generalize the argument of [CHT08, §2.4.4] to groups besides GL_n leads to a deformation condition based on parabolics which is *not* liftable. Instead, inspired by the arguments of [Tay08, §3] we define a deformation condition for symplectic and orthogonal groups based on deformations of a nilpotent element of $\mathfrak{g}_k = \mathrm{Lie}\, G_k$.

Let us first review the minimally ramified deformation condition introduced for GL_n in [CHT08, §2.4.4]. The first step is to reduce to studying certain tamely ramified representations. Recall that Γ_L^{t}, the Galois group of the maximal tamely ramified extension of L, is isomorphic to the semi-direct product

$$\widehat{\mathbf{Z}} \ltimes \prod_{p' \neq \ell} \mathbf{Z}_{p'}$$

where $\widehat{\mathbf{Z}}$ is generated by a Frobenius ϕ for L and the conjugation action by ϕ on each $\mathbf{Z}_{p'}$ is given by the p'-adic cyclotomic character. We consider tamely ramified representations which factor through the quotient $\widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$ (recall $p \neq \ell$). Picking a topological generator τ for $\mathbf{Z}_p$, the action is explicitly given by

$$\phi \tau \phi^{-1} = q\tau$$

where q is the size of the residue field of L. Note q is a power of ℓ, so it is relatively prime to p. Arguments in [CHT08] reduce the lifting problem to studying representations of the group $T_q := \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$. This reduction generalizes without surprises to symplectic and orthogonal groups.

The second step is to specify when a lift of $\overline{\rho} : T_q \to \mathrm{GL}_n(k)$ is "ramified no worse than $\overline{\rho}$". For suitable "coefficient rings" R, a deformation $\rho : T_q \to \mathrm{GL}_n(R)$ is *minimally ramified* according to [CHT08] when the natural k-linear map

$$\ker\left((\rho(\tau) - 1_n)^i\right) \otimes_R k \to \ker\left((\overline{\rho}(\tau) - 1_n)^i\right) \tag{1.2.3.1}$$

is an isomorphism for all i. The deformation condition is analyzed as follows:

- defining $V_i = \ker\left((\overline{\rho}(\tau) - 1_n)^i\right)$ gives a flag

  $$0 \subset V_r \subset V_{r-1} \subset \ldots \subset V_1 \subset k^n.$$

  This flag determines a parabolic k-subgroup $\overline{P} \subset \mathrm{GL}_n$ (points which preserve the flag) such that $\overline{\rho}(\tau) \in (\mathcal{R}_u \overline{P})(k)$ and $\overline{\rho}(\phi) \in \overline{P}(k)$;

- lift $\overline{P}$ to a parabolic subgroup P of GL_n. The deformation functor of such lifts is formally smooth, and for any minimally ramified deformation ρ over R there is a choice of such P for which $\rho(\tau) \in (\mathcal{R}_u P)(R)$ and $\rho(\phi) \in P(R)$. Conversely, any ρ with this property is minimally ramified;
- Finally, for the standard block-upper-triangular choice of P, one shows the deformation functor

$$\{(T, \Phi) : T \in \mathcal{R}_u P, \Phi \in P, \Phi T \Phi^{-1} = T^q, \overline{T} = \overline{\rho}(\tau), \overline{\Phi} = \overline{\rho}(\phi)\}$$

is formally smooth by building the universal lift over a power series ring: this uses explicit calculations with block-upper-triangular matrices.

To generalize beyond GL_n, we need to replace (1.2.3.1) with a more group-theoretic criterion. The naive generalization is to associate a parabolic $\overline{P}$ to $\overline{\rho}$ and then use the following definition.

Definition 1.2.3.1. For a "coefficient ring" R, say a lift $\rho : T_q \to G(R)$ is *ramified with respect to* $\overline{P}$ provided that there exists a parabolic R-subgroup $P \subset G_R$ lifting $\overline{P}$ such that $\rho(\tau) \in (\mathcal{R}_u P)(R)$ and $\rho(\phi) \in P(R)$.

This idea does not work. Let us focus on the symplectic case to illustrate what goes wrong.

The first problem is to associate a parabolic subgroup to $\overline{\rho}$. Recall that parabolic subgroups of a symplectic group correspond to isotropic flags $0 \subset V_1 \subset \ldots \subset V_r \subset V_r^{\perp} \subset \ldots \subset V_1^{\perp} \subset k^{2n}$. There is no reason that the flag determined by (1.2.3.1) is isotropic, so we would need some other method of producing a parabolic $\overline{P}$ such that $\overline{\rho}(\tau) \in (\mathcal{R}_u \overline{P})(k)$. In [BT71], Borel and Tits give a *natural* way to associate to the unipotent $\overline{\rho}(\tau)$ a smooth connected unipotent k-subgroup of G. The normalizer of this subgroup is always parabolic and so gives a candidate for $\overline{P}$. However, working out examples in GL_n for small n shows that this produces a different parabolic than the one determined by (1.2.3.1). This raises the natural question of how sensitive the smoothness of the deformation condition is to the choice of parabolic.

This leads to the second, larger problem: there are examples such that for *every* parabolic $\overline{P}$ satisfying $\overline{\rho}(\tau) \in (\mathcal{R}_u \overline{P})(k)$, not all deformations ramified with respect to $\overline{P}$ are liftable.

Example 1.2.3.2. Take $L = \mathbf{Q}_{29}$ and $k = \mathbf{F}_7$. Consider the representation $\overline{\rho} : T_{29} \simeq \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_7 \to \mathrm{GSp}_4(\mathbf{F}_7)$ defined by

$$\overline{\rho}(\tau) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \overline{\rho}(\phi) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The deformation condition of lifts ramified relative to any parabolic of GSp_4 whose unipotent radical contains $\overline{\rho}(\tau)$ is not liftable: there are lifts to the dual numbers that do not lift to $\mathbf{F}_7[\epsilon]/(\epsilon^3)$. This is easy to check with a computer algebra system such as [SAGE], since the existence of lifts can be reduced to a problem in linear algebra.

This latter problem is a general phenomenon, which we will explain conceptually in terms of algebraic geometry in §4.4.3.

The correct approach is to define a lift $\rho : T_q \to G(R)$ to be minimally ramified if $\rho(\tau)$ has "the same unipotent structure" as $\overline{\rho}(\tau)$. It is more convenient to work with nilpotent elements, using the exponential and logarithm maps (defined for nilpotent and unipotent elements since $p > n$). We wish to study lifts of the nilpotent $\overline{N} = \log(\overline{\rho}(\tau))$ to $N \in \mathfrak{g}$ that "remain nilpotent of the same nilpotent type as $\overline{N}$".

In Chapter 4, we make this notion of "same nilpotent type" rigorous for classical groups. There are combinatorial parametrization of nilpotent orbits of algebraic groups over an algebraically closed field, for example in terms of partitions or root data, which make precise the notion that the values of $N \in \mathfrak{g}_{\mathcal{O}}$ in the special and generic fiber lie in the "same" nilpotent orbit. For each nilpotent orbit σ (classified by purely combinatorial data, without reference to the field), pick an element $N_\sigma \in \mathfrak{g}_{\mathcal{O}}$ with this property lifting $\overline{N} \in \mathfrak{g}_k$. For a coefficient ring R, we define the "pure nilpotents" lifting $\overline{N}$ to be the $\widehat{G}(R)$-conjugates of N_σ.

Example 1.2.3.3. For example, let $G = \mathrm{GL}_3$ and

$$\overline{N} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Consider the lifts

$$N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g} \quad \text{and} \quad N_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}.$$

Both are nilpotent under the embedding of $\mathcal{O}$ into its fraction field K. The images of N_1 in $\mathfrak{g}_K$ and $\mathfrak{g}_k$ both lie in the nilpotent orbit corresponding to the partition $2+1$, so N_1 is an example of the type of nilpotent lift we want to consider. On the other hand, the image of N_2 in $\mathfrak{g}_K$ lies in the nilpotent orbit corresponding to the partition 3, so we do not want to use it. The pure nilpotents lifting $\overline{N}$ are $\widehat{G}(R)$-conjugates of N_1.

We then define a lift $\rho : T_q \to G(R)$ to be *minimally ramified* provided $\rho(\tau)$ is the exponential of a pure nilpotent lifting $\log \overline{\rho}(\tau) = \overline{N}$. In §4.4, we show that this deformation condition is liftable. The main technical fact needed to analyze this deformation condition is that the scheme-theoretic centralizer $Z_G(N_\sigma)$ is smooth over $\mathcal{O}$ for N_σ as above. The smoothness of such centralizers over algebraically closed fields is well-understood, and in §4.2 we study $Z_G(N_\sigma)$ and show that $Z_G(N_\sigma)$ is *flat* over $\mathcal{O}$ and hence smooth: Lemma 4.2.3.1 gives a criterion for flatness that is easy to verify for classical groups which suffices for our applications; there are difficulties beyond the classical case due the structure of $\pi_0(Z_G(N_\sigma)_{\overline{k}})$ in general.

Remark 1.2.3.4. It is a fortuitous coincidence (for [CHT08]) that for GL_n the lifts minimally ramified in the preceding sense are exactly the lifts ramified with respect to a parabolic subgroup of G. This rests on the fact that all nilpotent orbits of GL_n are Richardson orbits (see §4.4.3 for details).

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Chapter 2

Producing Geometric Deformations

In this chapter, we prove a generalization of Ramakrishna's lifting technique that applies to a wide class of reductive Chevalley groups G (Theorem 2.5.3.4). The key result is Proposition 2.5.2.1, a local-to-global principle for lifting Galois representations: under appropriate conditions, for a finite field k and a number field K with absolute Galois group Γ_K it is possible to lift a global representation $\rho : \Gamma_K \to G(R/I)$ through a square zero extension $R \to R/I$ provided it is possible to lift the local representations $\rho|_{\Gamma_v}$ for v in a specific (finite) set of places. This reduces the problem to studying local Galois deformation rings, which we do in subsequent chapters.

This technique was first used in [Ram99] and [Ram02] to produce geometric deformations for two-dimensional representations. The method for GL_2 was axiomatized by Taylor [Tay03], making it easier to generalize to other groups. We will generalize the technique to apply to any split connected reductive group, given as input local deformation conditions satisfying certain axioms; instances of such conditions are defined and studied in later chapters for symplectic and orthogonal groups. Patrikis has independently generalized this lifting technique [Pat15, Theorem 6.4]. The approach is the same, but he relies on different local deformation conditions.

Some background material about algebraic groups is reviewed in §2.1, in particular the notion of a very good prime and some facts about finite groups of Lie type. In §2.2, we review background on Galois cohomology and the deformation theory of Galois representations. Next in §2.3 we discuss a technical condition, bigness, on the residual representation necessary for the method to work, and show that if the image of $\overline{\rho}$ contains the k-points of the derived group then these conditions hold provided the characteristic of k is large enough relative to the Coxeter number of (the root system of) G. We construct an auxiliary local condition that Ramakrishna's method needs in §2.4. Finally in §2.5 we generalize Ramakrishna's method, providing a local to global principle after allowing the representation to ramify at finitely many additional places of K subject to this auxiliary condition.

2.1 Preliminaries about Algebraic Groups

This section collects some results about algebraic groups. In particular, we review the notion of a very good prime and we give a few results about finite groups of Lie type.

2.1.1 Good and Very Good Primes

Let p be a prime, Φ a reduced and irreducible root system, and $P = (\mathbf{Z}\Phi^\vee)^*$ the weight lattice for Φ. Suppose G is a connected reductive group over a field k.

Definition 2.1.1.1. The prime p is *good* for Φ provided that $\mathbf{Z}\Phi/\mathbf{Z}\Phi'$ is p-torsion free for all subsets $\Phi' \subset \Phi$. A good prime is *very good* provided that $P/\mathbf{Z}\Phi'$ is p-torsion free for all subsets $\Phi' \subset \Phi$. A prime is *bad* if it is not good.

Likewise, we say a prime p is *good* (or *very good*) for a general reduced root system if it is good (or very good) for each irreducible component. A prime p is *good* (or *very good*) for G provided it is good (or very good) for the root system of $G_{\overline{k}}$.

Let $\{\alpha_1, \dots, \alpha_r\}$ be a basis of positive roots for Φ. Let $\sum_i m_i \alpha_i$ be the highest root.

Fact 2.1.1.2. *The prime p is bad for Φ if and only if there is an i with $p|m_i$.*

This is [SS70, §I.4.3].

Example 2.1.1.3. Using the characterization in terms of the highest root, we obtain:

- For type A_n $(n \geq 1)$, all primes are good.
- For types B_n $(n \geq 2)$, C_n $(n \geq 2)$, or D_n $(n \geq 4)$, p is good if and only if $p \neq 2$.
- For types E_6, E_7, F_4 or G_2, p is good if and only if $p \neq 2, 3$.
- For type E_8, p is good if and only if $p \neq 2, 3, 5$.

Furthermore, p is very good if and only if p is good and moreover $p \nmid n+1$ when Φ is of type A_n due to:

Lemma 2.1.1.4. *The prime p is very good for Φ if and only if p is good for Φ and $p \nmid \#\pi_1(\Phi)$.*

Proof. For a subset $\Phi' \subset \Phi$, consider the exact sequence

$$0 \to \frac{\mathbf{Z}\Phi}{\mathbf{Z}\Phi'} \to \frac{P}{\mathbf{Z}\Phi'} \to \frac{P}{\mathbf{Z}\Phi} \to 0.$$

The right term is the definition of $\pi_1(\Phi)$. The middle term is p-torsion free if the left and right terms are, proving the "only if" statement. Conversely, if p is very good (the middle term is p-torsion free) by choosing $\Phi' = \Phi$ we get $p \nmid \#\pi_1(\Phi)$. ☐

We record some consequences of p being a very good prime. Let G' be the derived group of G, $\mathfrak{g} = \operatorname{Lie} G$, and $\mathfrak{g}' = \operatorname{Lie} G'$. Denote the center of G by Z_G. Suppose that G is k-split.

Lemma 2.1.1.5. *If p is very good for G, the order of the fundamental group of G' and the order of the center of G' are prime to p. Furthermore, any central isogeny $H \to G'$ or $G' \to H$ for a connected semisimple k-group H induces an isomorphism on Lie algebras.*

Proof. As p is very good, the central isogeny from the simply connected cover of G' to the adjoint quotient of G' has order prime to p: the degree is the size of the fundamental group of the root system. Thus the kernel of a central isogeny between G' and H has order to prime to p, so it is étale and its Lie algebra is zero. ☐

Lemma 2.1.1.6. *Let p be very good for G. Then $\operatorname{Lie}(Z_G)$ is the center $\mathfrak{z}_{\mathfrak{g}}$ of $\mathfrak{g}$ and moreover $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}} \oplus \mathfrak{g}'$.*

Proof. Consider the adjoint map $\operatorname{Ad}_G : G \to \operatorname{GL}(\mathfrak{g})$. Its scheme theoretic kernel is Z_G, so the kernel of $\operatorname{Lie}(\operatorname{Ad}_{\mathfrak{g}}) : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is $\operatorname{Lie} Z_G$. But $\operatorname{Lie}(\operatorname{Ad}_G) = \operatorname{ad}_{\mathfrak{g}}$, the kernel of which is $\mathfrak{z}_{\mathfrak{g}}$. As p is very good, the Lie algebra $\mathfrak{g}'$ maps isomorphically to the Lie algebra of the adjoint quotient G/Z_G by Lemma 2.1.1.5. This gives the decomposition. ☐

2.1.2 Finite Groups of Lie Type

Now suppose k is finite, with $q = \#k$. Continue to assume the characteristic p of k is very good for G. We record a couple of classical facts from the theory of finite groups of Lie type.

Lemma 2.1.2.1. *If G is a split, absolutely simple, connected semisimple group over k, $\mathfrak{g} = \operatorname{Lie} G$ is an absolutely irreducible $G(k)$-module provided that $q > 3$.*

Proof. The adjoint representation $\operatorname{Ad}_G : G_{\overline{k}} \to \operatorname{GL}(\mathfrak{g})$ is an absolutely irreducible representation of algebraic groups as p is very good [MT11, Theorem 15.20]. Its highest weight is the highest root $\lambda = \sum_i m_i \alpha_i$. If $\langle \lambda, \alpha_i^\vee \rangle < q$ for all α_i, then the adjoint representation will be absolutely irreducible as a $G(k)$-module; this was originally proven in [Ste63], with a modern proof in [Hum06, §2.12]. The value of $\langle \lambda, \alpha_i^\vee \rangle$ can be read off from the affine Dynkin diagram of the root system, as the additional node is the lowest root. In particular, it is at most 3. ☐

Lemma 2.1.2.2. *Suppose that $q > 3$. Let G be a split connected semsimple group over k with simply connected central cover $\pi : \widetilde{G} \to G$. Then $[G(k), G(k)] = \pi(\widetilde{G}(k))$.*

Proof. Consider the exact sequence

$$1 \to (\ker \pi)(k) \to \widetilde{G'}(k) \to G'(k) \to H^1(k, \ker \pi)$$

(using fppf H^1). We know that $\widetilde{G'}(k)$ is perfect: $\widetilde{G'}$ is a product of absolutely simple semisimple simply connected groups, and the k-points of all such groups, with a few exceptions ruled out by the assumption $q > 3$, are perfect [MT11, Theorem 24.11]. Thus $\pi(\widetilde{G'}(k)) \subset [G'(k), G'(k)]$. As $G'(k)/\pi(\widetilde{G'}(k)) \subset H^1(k, \ker \pi)$ is abelian, we conclude that $\pi(\widetilde{G'}(k)) = [G'(k), G'(k)]$. □

Remark 2.1.2.3. While G is always perfect as an algebraic group, $G(k)$ need not be perfect. For example, consider $G = \mathrm{PGL}_n$, which is perfect as an algebraic group. One can calculate that $[\mathrm{PGL}_n(k), \mathrm{PGL}_n(k)] \simeq \mathrm{SL}_n(k)/\mu_n(k)$ which has index $\gcd(n, q-1)$ in $\mathrm{PGL}_n(k)$.

Now we record and slightly generalize some results about the group cohomology from [CPS75] for G a split connected semisimple group over k.

Proposition 2.1.2.4. *If $p \neq 2$ is very good for G and $q \neq 2, 3, 4, 5, 9$ then $H^1(G(k), \mathfrak{g}) = 0$.*

Before we prove this result, we need some preparation. Let Δ be a set of positive simple roots corresponding to a Borel $B \subset G$ containing a split maximal torus T. Denote the unipotent radical of B by U. Consider $V = \mathfrak{g}$ as a representation of $G(k)$. Roots $\alpha \in \Phi(G, T)$ give homomorphisms $\alpha_k : T(k) \to k^\times$. We say that two roots α and β are *equivalent over k* if $\ker \alpha_k = \ker \beta_k$. We say that α is *equivalent over k to 0* if $\alpha_k = 1$.

Remark 2.1.2.5. This notion is called "Galois equivalence" in [CPS75]. When $q \neq 2$ (so $k^\times \neq 1$), such equivalences occur only for roots in a common irreducible component of $\Phi(G, T)$, or between a root and zero, and occur only over small fields [CPS75, Proposition 3.3]: none are possible when $q \neq 2, 3, 4, 5, 9$.

For the rest of §2.1.2, we assume that p and $q = \#k$ are as in Proposition 2.1.2.4.

Lemma 2.1.2.6. *We have $V^{B(k)} = 0$.*

Proof. As the characteristic is very good, it suffices to prove this for the simply connected central cover $\pi : \widetilde{G} \to G$: the Lie algebra is unchanged, and $\pi^{-1}(B)$ is a Borel subgroup of $\widetilde{G}$ containing a split maximal torus $\pi^{-1}(T)$. As G is now simply connected, $\mathfrak{t} := \operatorname{Lie} T$ decomposes as

$$\mathfrak{t} = \bigoplus_{\alpha \in \Delta} \mathfrak{t}_{\alpha^\vee}$$

where $\mathfrak{t}_{\alpha^\vee}$ is the coroot line associated to the simple coroot $\alpha^\vee$. As $q \neq 2, 3$, $\alpha_k : T(k) \to k^\times$ is non-trivial for every root α by [CPS75, Proposition 3.1]. Hence $V^{T(k)} \subset \mathfrak{t}$, so $V^{B(k)} \subset \mathfrak{t}$.

Let $t_\alpha = d\alpha^\vee(x^{-1}\partial_x|_{x=1})$ be the standard basis vector for $\mathfrak{t}_{\alpha^\vee}$. Consider an element $v \in \mathfrak{t}$ fixed by $B(k)$, and decompose it as

$$v = \sum_{\alpha \in \Delta} v_\alpha t_\alpha$$

with $v_\alpha \in k$. For $\beta \in \Delta$, fix an isomorphism $u_\beta : \mathbf{G}_a \simeq U_\beta$ so $e_\beta = du_\beta(\partial_x|_{x=0})$. For $u = u_\beta(1) \in U_\beta(k)$, we wish to calculate

$$\mathrm{Ad}(u)(v) = \sum_{\alpha \in \Delta} v_\alpha \, \mathrm{ad}(u) t_\alpha.$$

We calculate that for $z \in \mathbf{G}_m$,

$$\begin{aligned} u\alpha^\vee(z)u^{-1} &= \alpha^\vee(z) \left(\alpha^\vee(z)^{-1} u_\beta(1) \alpha^\vee(z)\right) u_\beta(1)^{-1} \\ &= \alpha^\vee(z) u_\beta\left(z^{-\langle \beta, \alpha^\vee \rangle} - 1\right). \end{aligned}$$

Thus we obtain

$$\mathrm{Ad}(u)t_\alpha = t_\alpha - \langle \beta, \alpha^\vee \rangle \cdot e_\beta.$$

Applying $\sum_{\alpha\in\Delta} v_\alpha(\cdot)$ to both sides gives

$$\mathrm{Ad}(u)v = v - \left(\sum_{\alpha\in\Delta} v_\alpha \langle \beta, \alpha^\vee \rangle v_\alpha \right) e_\beta.$$

If v is fixed by $U_\beta(k)$, hence by $u = u_\beta(1)$, it follows that

$$\sum_{\alpha\in\Delta} v_\alpha \langle \beta, \alpha^\vee \rangle = 0$$

since $e_\beta \in \mathfrak{g}$ is non-zero. If this holds for all $\beta \in \Delta$, then the Δ-tuple $(v_\alpha) \in k^\Delta$ is in the kernel of the Cartan matrix over k. But the Cartan matrix is invertible over k as the determinant is a unit in very good characteristic. (This can be verified by inspecting tables of root systems.) Hence $V^{B(k)} = 0$. □

Now we prove Proposition 2.1.2.4. In the case that G is simply connected, we may apply [CPS75, Corollary 2.9]: for the adjoint representation $V = \mathfrak{g}$ of $G(k)$, letting $V[\alpha]$ denote the weight space for α_k, it states

$$\dim_k H^1(G(k), V) = \left(\sum_{\alpha\in\Delta} \dim_k V[\alpha] \right) - \dim_k V[0],$$

as there are no equivalences between distinct roots, nor between roots and 0, because of the restrictions on q. The weight space for $\alpha \in \Delta$ (as representations of $T(k)$) is therefore the α-root line. Since $\#\Delta = \dim T$ and $\mathrm{Lie}\, T = V[0]$, we obtain that $H^1(G(k), \mathfrak{g}) = 0$.

For general G, let $\pi : \widetilde{G} \to G$ be the simply connected central cover of G. The central k-subgroup scheme $\ker \pi$ is finite, of order prime to p as p is very good, and $d\pi : \mathrm{Lie}\, \widetilde{G} \to \mathfrak{g}$ is an isomorphism. One instance of the inflation-restriction sequence reads

$$0 \to H^1(\pi(\widetilde{G}(k)), \mathfrak{g}) \to H^1(\widetilde{G}(k), \mathfrak{g})$$

As the final term is 0, so is the middle term. Now $\pi(\widetilde{G}(k))$ has index prime to p in $G(k)$ as the index divides $\#H^1(k, \ker \alpha)$, so the composition

$$\mathrm{cor} \circ \mathrm{res} : H^1(G(k), \mathfrak{g}) \to H^1(G(k), \mathfrak{g})$$

is an isomorphism. Since $H^1(\pi(\widetilde{G}(k)), \mathfrak{g}) = 0$, it follows that $H^1(G(k), \mathfrak{g}) = 0$ as desired. This completes the proof of Proposition 2.1.2.4. □

2.2 Review of Galois Cohomology and Deformations

2.2.1 Results about Galois Cohomology

Let K_v be a p-adic field with normalized valuation v and absolute Galois group Γ_{K_v}, and let V be a finite discrete Γ_{K_v}-module.. Let μ_∞ denote the Galois module of roots of unity in $\overline{K_v}$. Define the (Cartier) dual of V to be

$$V^* := \mathrm{Hom}_{\mathbf{Z}}(V, \mu_\infty).$$

As a Galois representation it is a Tate twist of the $\mathbf{Q}/\mathbf{Z}$ dual of V. There is a natural evaluation pairing $V \otimes V^* \to \mu_\infty$ as Γ_{K_v}-modules. We recall some standard facts about Galois cohomology for local and global fields (found for example in [NSW08, Chapter VII, VIII]). Let $H^i(K_v, V) := H^i(\Gamma_{K_v}, V)$ be the ith (continuous) Galois cohomology group.

Fact 2.2.1.1 (Local Tate Duality)**.** *For $i = 0, 1, 2$ the cup product gives a non-degenerate pairing*

$$H^i(K_v, V) \times H^{2-i}(K_v, V^*) \to H^2(K_v, \mu_\infty) \simeq \mathbf{Q}/\mathbf{Z}.$$

Fact 2.2.1.2 (Local Euler-Poincaré Characteristic)**.** *Let* $N = \|\#V\|_v$ *(the normalized absolute value, or equivalently the index of* $(\#V)\mathcal{O}_{K_v}$ *in* $\mathcal{O}_{K_v}$*). Then*

$$\frac{\#H^0(K_v,V)\#H^2(K_v,V)}{\#H^1(K_v,V)} = \frac{1}{N}$$

Let I_{K_v} be the inertia subgroup of Γ_{K_v}.

Definition 2.2.1.3. The *unramified* ith cohomology of V is the group

$$H^i_{\mathrm{nr}}(K_v,V) := \mathrm{image}\left(H^i(\Gamma_{K_v}/I_{K_v}, V^{I_{K_v}}) \to H^i(K_v,V)\right).$$

Fact 2.2.1.4. *We have* $H^1_{\mathrm{nr}}(K_v,V) = \ker\left(H^1(K_v,V) \to H^1(I_{K_v},V)\right)$ *and* $\#H^1_{\mathrm{nr}}(K_v,V) = \#H^0(K_v,V)$*.*

Now let K be a number field with absolute Galois group Γ_K, and let S be a finite set of places of K containing the archimedean places. Let K_S be the maximal extension of K unramified outside of S and $\Gamma_S = \mathrm{Gal}(K_S/K)$. If v is a place of K, denote the decomposition group at v in Γ_K by Γ_v (well-defined up to conjugation); here we allow $v|\infty$. For a finite discrete Γ_S-module W and $v \in S$, there is a restriction map $\mathrm{res}_v : H^i(\Gamma_S, W) \to H^i(K_v, W)$ where $H^i(K_v,W) := H^i(\Gamma_v, W)$.

Definition 2.2.1.5. For a place $v \in S$ (allowing $v|\infty$), a *local condition* N for W at v is a subgroup of $H^1(K_v,W)$. A *global condition* N_S for W is a collection of local conditions N_v, one for each $v \in S$. The *generalized Selmer group* for W with respect to N_S is

$$H^1_{N_S}(\Gamma_S,W) := \{x \in H^1(\Gamma_S,W) |\, \mathrm{res}_v(x) \in N_v \text{ for all } v \in S\}.$$

Given a local condition N at non-archimedean v, one can define a *dual* local condition $N^\perp \subset H^1(K_v,W^*)$: it is the exact annihilator of N via Tate local duality. There is also a version for achimedean places [NSW08, Theorem 7.2.17]. Then one defines a dual global condition $N_S^\perp$ to be the collection $\{N_v^\perp\}$ for places v of S.

Remark 2.2.1.6. Let W be a finite discrete Γ_K-module that is unramified outside of S. We often think of a global condition as being a collection of local conditions for *every* place of K, where for $v \notin S$ the deformation condition is the unramified condition $H^1_{\mathrm{nr}}(K_v,W)$. This allows us to work with Γ_K instead of Γ_S, which will be convenient when we want to enlarge S later. If the global condition N_S is the unramified condition outside of S, we will write

$$H^1_{N_S}(K,W) := \{x \in H^1(K,W) |\, \mathrm{res}_v(x) \in N_v \text{ for all } v\}.$$

Unwinding definitions, it follows from Fact 2.2.1.4 that

$$H^1_{N_S}(\Gamma_S,W) = H^1_{N_S}(K,W).$$

This convention behaves well with respect to the notion of dual global conditions because when $v \nmid \infty$ and $\#W$ is a v-unit (as holds for all but finitely many v), the groups $H^1_{\mathrm{nr}}(\Gamma_v,W)$ and $H^1_{\mathrm{nr}}(\Gamma_v,W^*)$ are exact annihilators under the Tate-local duality pairing.

The generalized Selmer groups as in Definition 2.2.1.5 fit into a long exact sequence, known as the Poitou-Tate exact sequence. For us, the relevant part is the following five term sequence.

Fact 2.2.1.7 (Modified Poitou-Tate Exact Sequence)**.** *If* $\#W$ *is an* S*-unit, there is an exact sequence*

$$H^1(\Gamma_S,W) \to \bigoplus_{v\in S} H^1(K_v,W)/N_v \to H^1_{N_S^\perp}(K,W^*)^\vee \to H^2(\Gamma_S,W) \to \bigoplus_{v\in S} H^2(K_v,W).$$

This gives the following equality due to Wiles [NSW08, Theorem 8.7.9]:

$$\frac{\#H^1_{N_S}(K,W)}{\#H^1_{N_S^\perp}(K,W^*)} = \frac{\#H^0(K,W)}{\#H^0(K,W^*)} \prod_{v\in S} \frac{\#N_v}{\#H^0(K_v,W)}. \tag{2.2.1.1}$$

Note that $\#N_v = \#H^0(K_v,W)$ for the unramified condition at $v \in S$ by Fact 2.2.1.4, so the product is insensitive to enlarging S provided the additional local conditions are the unramified condition.

2.2.2 Deformations of Galois Representations

Next we recall some facts about the deformation theory for Galois representations: a basic reference is [Maz97], with the extension to algebraic groups beyond GL_n discussed in [Til96].

Let k be a finite field of characteristic p, and consider the category of complete local Noetherian rings with residue field k, where morphisms are local homomorphisms that induce the identity map on k. Objects of this category are called coefficient rings. For a coefficient ring $\mathcal{O}$, a *coefficient $\mathcal{O}$-algebra* is a coefficient ring which is a $\mathcal{O}$-algebra such that the structure morphism is a map of coefficient rings. Denote the category of coefficient $\mathcal{O}$-algebras by $\widehat{\mathcal{C}}_{\mathcal{O}}$, and the full subcategory of Artinian coefficient $\mathcal{O}$-algebras by $\mathcal{C}_{\mathcal{O}}$. Note that all coefficient rings are coefficient $W(k)$-algebras.

Definition 2.2.2.1. A *small* surjection of coefficient $\mathcal{O}$-algebras $f : A_1 \to A_0$ is a surjection such that $\ker(f) \cdot \mathfrak{m}_{A_1} = 0$.

Let Γ be a pro-finite group satisfying the following finiteness property: for every open subgroup $\Gamma_0 \subset \Gamma$, there are only finitely many continuous homomorphisms from Γ_0 to $\mathbf{Z}/p\mathbf{Z}$. This is true for the absolute Galois group of a local field and for the Galois group of the maximal extension of a number field unramified outside a finite set of places.

Consider a reductive group scheme G (with connected fibers) over a coefficient ring $\mathcal{O}$ with derived group G' whose center is smooth over $\mathcal{O}$. Assume that p is very good for G_k (in the sense of Definition 2.1.1.1). For $A \in \widehat{\mathcal{C}}_{\mathcal{O}}$, define

$$\widehat{G}(A) := \ker(G(A) \to G(k))$$

We are interested in deforming a fixed $\overline{\rho} : \Gamma \to G(k)$. Let $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathfrak{g}' = \mathrm{Lie}\, G'$.

- Let $f : A_1 \to A_0$ be a morphism in $\widehat{\mathcal{C}}_{\mathcal{O}}$ and $\rho_0 : \Gamma \to G(A_0)$ a continuous homomorphism. A *lift* of ρ_0 to A_1 is a continuous homomorphism $\rho_1 : \Gamma \to G(A_1)$ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\rho_1} & G(A_1) \\ & \searrow^{\rho_0} & \downarrow f \\ & & G(A_0) \end{array}$$

  Define the functor $D^{\square}_{\overline{\rho},\mathcal{O}} : \widehat{\mathcal{C}}_{\mathcal{O}} \to \mathrm{Sets}$ by sending a coefficient $\mathcal{O}$-algebra A to the set of lifts of $\overline{\rho}$ to A.

- With the notation as before, two lifts ρ and ρ' of $\overline{\rho}$ to $A_1 \in \mathcal{C}_{\mathcal{O}}$ are *strictly equivalent* if they are conjugate by an element of $\widehat{G}(A_1)$. A *deformation* of ρ_0 to A_1 is a strict equivalence class of lifts. Define the functor $D_{\overline{\rho},\mathcal{O}} : \widehat{\mathcal{C}}_{\mathcal{O}} \to \mathrm{Sets}$ by sending a coefficient $\mathcal{O}$-algebra A to the set of deformations of $\overline{\rho}$ to A.

We will drop the subscript $\mathcal{O}$ when it is clear from context.

Fact 2.2.2.2. *The functor $D^{\square}_{\overline{\rho},\mathcal{O}}$ is representable. When $\mathfrak{g}_k^{\Gamma} = \mathrm{Lie}(Z_G)_k$, the functor $D_{\overline{\rho},\mathcal{O}}$ is representable.*

Remark 2.2.2.3. The first part is simple, the second is a reformulation of [Til96, Theorem 3.3].

The representing objects are denoted $R^{\square}_{\overline{\rho},\mathcal{O}}$ and (when it exists) $R_{\overline{\rho},\mathcal{O}}$. While we usually care about deformations, it is technically easier to work with lifts.

Example 2.2.2.4. For $G = \mathrm{GL}_n$, a homomorphism $\rho : \Gamma \to G(A)$ is the data of an $A[\Gamma]$-module which is a free A-module of rank n (on which the Γ action is continuous) together with a basis. Deformations forget the basis. The deformation functor is representable, for example, when $\overline{\rho}$ is absolutely irreducible, as by Schur's lemma only scalar matrices commute with $\overline{\rho}$.

This deformation theory is controlled by Galois cohomology. Let $\mathrm{ad}(\overline{\rho})$ denote the representation of Γ on $\mathfrak{g}_k$ via the adjoint representation. We also consider the representation $\mathrm{ad}^0(\overline{\rho})$ of Γ on $\mathfrak{g}'_k$. By Corollary 2.1.1.6, as p is very good $\mathfrak{g}_k = \mathfrak{g}'_k \oplus \mathfrak{z}_{\mathfrak{g}}$, where $\mathfrak{z}_{\mathfrak{g}}$ is the Lie algebra of Z_G. The condition in Fact 2.2.2.2 is just that

$H^0(\Gamma, \mathrm{ad}(\overline{\rho})) = \mathfrak{z}_{\mathfrak{g}}$, or equivalently that $H^0(\Gamma, \mathrm{ad}^0(\overline{\rho})) = 0$. In general, since p is very good the natural map $H^i(\Gamma, \mathrm{ad}^0(\overline{\rho})) \to H^i(\Gamma, \mathrm{ad}(\overline{\rho}))$ is injective for all i; we often use this without comment.

The degree-1 Galois cohomology groups are related to tangent spaces. When $D_{\overline{\rho},\mathcal{O}}$ is representable, the tangent space to $R_{\overline{\rho},\mathcal{O}}$ can be identified with $D_{\overline{\rho},\mathcal{O}}(k[\epsilon]/\epsilon^2)$. The latter makes sense even when $D_{\overline{\rho},\mathcal{O}}$ is not representable. The tangent spaces can be analyzed by the first order exponential map [Til96, §3.5]. For a smooth $\mathcal{O}$-group scheme G, and a small surjection $f : A \to A/I$ of coefficient rings ($I \cdot \mathfrak{m}_A = 0$), smoothness gives an isomorphism

$$\exp : \mathfrak{g} \otimes_k I \simeq \ker(G(A) \to G(A/I)) = \ker(\widehat{G}(A) \to \widehat{G}(A/I)).$$

Definition 2.2.2.5. For a smooth $\mathcal{O}$-group scheme G, and a small surjection $f : A \to A/I$ of coefficient rings ($I \cdot \mathfrak{m}_A = 0$), the *exponential map* is the induced map

$$\exp : \mathfrak{g} \otimes_k I \to \widehat{G}(A).$$

This is functorial in the $\mathcal{O}$-group G.

The tangent space of $D_{\overline{\rho},\mathcal{O}}$ is identified with $H^1(\Gamma, \mathrm{ad}(\overline{\rho}))$. Under this isomorphism, the cohomology class of a 1-cocycle τ corresponds to the lift $\rho(g) = \exp(\epsilon\tau(g))\overline{\rho}(g)$.

Remark 2.2.2.6. For the framed deformation ring $R^{\square}_{\overline{\rho},\mathcal{O}}$, the tangent space is identified with the k-vector space $Z^1(\Gamma, \mathrm{ad}(\overline{\rho}))$ of (continuous) 1-cocycles of Γ valued in $\mathrm{ad}(\overline{\rho})$. We also observe that

$$\dim_k Z^1(\Gamma, \mathrm{ad}(\overline{\rho})) - \dim_k H^1(\Gamma, \mathrm{ad}(\overline{\rho})) = \dim_k B^1(\Gamma, \mathrm{ad}(\overline{\rho})) = \dim_k \mathfrak{g} - \dim_k H^0(\Gamma, \mathrm{ad}(\overline{\rho}))$$

since the space of coboundaries admits a surjection from $\mathrm{ad}(\overline{\rho})$ with kernel $\mathrm{ad}(\overline{\rho})^{\Gamma}$. This will be useful when comparing dimensions of framed and unframed deformation rings that are smooth.

We will want to studying special classes of deformations. We work with the category $\mathcal{C}_{\mathcal{O}}$ of Artinian coefficient rings.

Definition 2.2.2.7. A *lifting condition* is a sub-functor $\mathcal{D}^{\square} \subset D^{\square}_{\overline{\rho},\mathcal{O}} : \mathcal{C}_{\mathcal{O}} \to \mathrm{Sets}$ such that:

1. For any coefficient ring A, $\mathcal{D}^{\square}(A)$ is closed under strict equivalence.
2. Given a Cartesian diagram in $\mathcal{C}_{\mathcal{O}}$

$$\begin{array}{ccc} A_1 \times_{A_0} A_2 & \xrightarrow{\pi_2} & A_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ A_1 & \longrightarrow & A_0 \end{array}$$

   and $\rho \in D^{\square}_{\overline{\rho},\mathcal{O}}(A_1 \times_{A_0} A_2)$, we have $\rho \in \mathcal{D}^{\square}(A_1 \times_{A_0} A_2)$ if and only if $\mathcal{D}^{\square}(\pi_1) \circ \rho \in \mathcal{D}^{\square}(A_1)$ and $\mathcal{D}^{\square}(\pi_2) \circ \rho \in \mathcal{D}^{\square}(A_2)$.

As it is closed under strict equivalence, we naturally obtain a *deformation condition*, a sub-functor $\mathcal{D} \subset D_{\overline{\rho},\mathcal{O}}$.

By Schlessinger's criterion [Sch68, Theorem 2.11] being a lifting condition is equivalent to the functor $\mathcal{D}^{\square}$ being pro-representable. (This is easy to check using that $\mathcal{D}^{\square}$ is a subfunctor of a representable functor.) Likewise, the deformation condition $\mathcal{D}$ associated to a lifting condition $\mathcal{D}^{\square}$ is pro-representable provided that $D_{\overline{\rho},\mathcal{O}}$ is.

Remark 2.2.2.8. To apply Schlessinger's criterion, we use the category $\mathcal{C}_{\mathcal{O}}$ of Artinian coefficient rings. Often the functor $\mathcal{D}$ could equally well be defined on the larger category $\widehat{\mathcal{C}}_{\mathcal{O}}$. Alternately, we can try to extend the functor $\mathcal{D}$ to $\widehat{\mathcal{C}}_{\mathcal{O}}$ by the definition $\mathcal{D}(A) = \varprojlim \mathcal{D}(A/\mathfrak{m}_A^n)$. It is sometimes subtle to check that this latter definition has an "expected" concrete meaning. For example, consider the case of torsion-crystalline representations: the fact that the inverse limit of torsion crystalline representations is actually a subquotient of a lattice in a crystalline representation was a conjecture of Fontaine proved by Liu [Liu07].

The tangent space of a deformation condition $\mathcal{D}$ is a k-subspace of $H^1(\Gamma, \mathrm{ad}(\overline{\rho}))$, and will be denoted by $H^1_{\mathcal{D}}(\Gamma, \mathrm{ad}(\overline{\rho}))$. For a small surjection $A_1 \to A_0$ and $\rho \in \mathcal{D}(A_0)$, the set of deformations of ρ to A_1 subject to $\mathcal{D}$ is a $H^1_{\mathcal{D}}(\Gamma, \mathrm{ad}(\overline{\rho}))$-torsor. This torsor-structure is compatible with the action of the unrestricted tangent space to $D_{\overline{\rho}}$ on the space of all deformations of ρ to A_1.

Example 2.2.2.9. Let G' be the derived group of G. The most basic examples of deformation conditions are the conditions imposed by fixing the lift of the homomorphism $\Gamma \to (G/G')(k)$. To be precise, for the quotient map $\mu : G \to G/G' =: S$, a fixed $\nu : \Gamma \to S(\mathcal{O})$ lifting $\mu \circ \overline{\rho}$, and $A \in \widehat{\mathcal{C}}_{\mathcal{O}}$ with structure morphism $\imath : \mathcal{O} \to A$, we define a deformation condition $\mathcal{D}_0 \subset \mathcal{D}_{\overline{\rho}}$ by

$$\mathcal{D}_\nu(A) = \{\rho \in \mathcal{D}_{\overline{\rho}}(A) | \Gamma \to G(A) : \mu_A \circ \rho = \imath \circ \nu_A\}.$$

One checks this is a deformation condition. Its tangent space is $H^1(\Gamma, \mathrm{ad}^0(\overline{\rho}))$ since p is very good. We define $\mathcal{D}^{\square}_{\overline{\rho}}$ similarly. We will use without comment that $(G/G')(A) = G(A)/G'(A)$ due to Lang's theorem since k is finite and G' is smooth over $\mathcal{O}$ with G'_k connected.

Another important easy example is the *unramified* deformation condition for any non-archimedean place v where ρ is unramified: this consists of lifts that are unramified (respectively, also with a specified choice of ν). The tangent space is $H^1_{\mathrm{nr}}(\Gamma_v, \mathrm{ad}(\overline{\rho}))$ (respectively $H^1_{\mathrm{nr}}(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))$).

For a small surjection of coefficient $\mathcal{O}$-algebras $f : A_1 \to A_0$ with kernel I, by using continuous cocycles we can define an obstruction $\mathrm{ob}\,\rho_0 \in H^2(\Gamma, \mathrm{ad}(\overline{\rho})) \otimes I$ to lifting.

Fact 2.2.2.10. *The representation ρ_0 lifts to A_1 if and only if* $\mathrm{ob}\,\rho_0 = 0$. *When a lift exists, the set of lifts of ρ_0 is naturally an $H^1(\Gamma, \mathrm{ad}(\overline{\rho})) \otimes I$-torsor.*

Definition 2.2.2.11. A deformation condition $\mathcal{D}$ is *locally liftable* (over $\mathcal{O}$) if for all small surjections $f : A_1 \to A_0$ of coefficient $\mathcal{O}$-algebras the natural map

$$\mathcal{D}(f) : \mathcal{D}(A_1) \to \mathcal{D}(A_0)$$

is surjective.

This holds, for example, if $H^2(\Gamma, \mathrm{ad}(\overline{\rho})) = 0$. A geometric way to check local liftability is to show that the corresponding deformation ring (when it exists) is smooth. Obviously it suffices to check liftability for lifts instead of deformations, so we can work with the framed deformation ring and avoid representability issues for $\mathcal{D}_{\overline{\rho}}$.

Example 2.2.2.12. The unramified deformation condition is liftable: an unramified lift is completely determined by the image of Frobenius in $G(A_0)$, and G is smooth over $\mathcal{O}$.

When attempting to lift with a fixed lift ν of $\Gamma \to (G/G')(k)$, the obstruction cocycle will lie in the group $H^2(\Gamma, \mathrm{ad}^0(\overline{\rho}))$. To see this, recall that the obstruction cocycle is defined by picking a set theoretic lift ρ_1 of a given $\rho_0 : \Gamma_K \to G(A_0)$: the 2-cocycle records the failure of ρ_1 to be a homomorphism. By choosing the continuous set-theoretic lift $\Gamma_K \to G(A_1)$ so that $\Gamma_K \to (G/G')(A_0)$ agrees with ν (as we may easily do since $\ker \rho_0$ is open in Γ_K), the obstruction cocycle clearly takes values in $\mathrm{ad}^0(\overline{\rho})$.

We now consider global deformation conditions. Let K be a number field, S a finite set of places of K that contains all the places of K at which $\overline{\rho}$ are ramified and all archimedean places. As before, let Γ_S be the Galois group of the maximal extension of K unramified outside of S and Γ_K be the absolute Galois group of K.

Definition 2.2.2.13. A *global deformation condition* $\mathcal{D}_S$ for $\overline{\rho} : \Gamma_S \to G(k)$ is a collection of local deformation conditions $\{\mathcal{D}_v\}_{v \in S}$ for $\overline{\rho}|_{\Gamma_v}$. We say it is *locally liftable* (over $\mathcal{O}$) if each $\mathcal{D}_v$ is locally liftable (over $\mathcal{O}$). A *global deformation of* $\overline{\rho} : \Gamma_S \to G(k)$ *subject to* $\mathcal{D}_S$ is a deformation $\rho : \Gamma_S \to G(A)$ such that $\rho|_{\Gamma_v} \in \mathcal{D}_v(A)$ for all $v \in S$.

Remark 2.2.2.14. Equivalently, a global deformation condition consists of a collection of local deformation conditions for every place of K such that $\mathcal{D}_v$ is the unramified condition for $v \notin S$, in which case we may work in terms of deformations of $\overline{\rho} : \Gamma_K \to G(k)$.

For $v \in S$, let L_v denote the tangent space of the local deformation condition $\mathcal{D}_v$. A global deformation condition gives a generalized Selmer group. We will be mainly interested in the *dual Selmer group*

$$H^1_{\mathcal{D}_S^\perp}(\Gamma_S, \mathrm{ad}(\overline{\rho})^*) = \{x \in H^1(\Gamma_S, \mathrm{ad}(\overline{\rho})^*) : \mathrm{res}_v(x) \in L_v^\perp \text{ for all } v \in S\}. \tag{2.2.2.1}$$

For Ramakrishna's method to work, it is crucial that the local tangent spaces be large enough relative to the local invariants. We say that a global deformation condition satisfies the *tangent space inequality* if

$$\sum_{v \in S} \dim L_v \geq \sum_{v \in S} \dim H^0(\Gamma_v, \mathrm{ad}^0(\overline{\rho})). \tag{2.2.2.2}$$

Remark 2.2.2.15. For later use, we define $h_v^i := \dim_k H^i(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))$ and $h_v^{*,i} := \dim_k H^i(\Gamma_v, \mathrm{ad}^0(\overline{\rho})^*)$.

2.3 Big Representations

Let $\mathcal{O}$ be the ring of integers in a p-adic field with residue field k, and let $q = \#k$. Consider a split connected reductive group scheme G over $\mathcal{O}$, with derived group G', and define $\mathfrak{g}$ and $\mathfrak{g}'$ to be the Lie algebras of G and G' respectively. Fix a split maximal $\mathcal{O}$-torus T of G. Let K be a number field and χ denote the p-adic cyclotomic character $\chi : \Gamma_K \to \mathbf{Z}_p^\times$, with reduction $\overline{\chi} : \Gamma_K \to \mathbf{F}_p^\times$.

2.3.1 Big Representations

The natural class of representations $\overline{\rho} : \Gamma_K \to G(k)$ to which Ramakrishna's method will apply are those which satisfy the following conditions:

Definition 2.3.1.1. A *big representation* $\overline{\rho} : \Gamma_K \to G(k)$ is a continuous homomorphism such that

(i) $\mathrm{ad}^0(\overline{\rho})$ is an absolutely irreducible representation of Γ_K;

(ii) letting $K(\mathrm{ad}^0(\overline{\rho}))$ (respectively $K(\mathrm{ad}^0(\overline{\rho})^*)$) denote the fixed field of the kernel of the action of Γ_K on $\mathrm{ad}^0(\overline{\rho})$ (respectively on $\mathrm{ad}^0(\overline{\rho})^*$), we have

$$H^1(\mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/K), \mathrm{ad}^0(\overline{\rho})) = 0 \quad \text{and} \quad H^1(\mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho})^*)/K), \mathrm{ad}^0(\overline{\rho})^*) = 0;$$

(iii) there exists $\gamma \in \Gamma_K$ such that $\overline{\rho}(\gamma)$ is regular semisimple with associated maximal torus $Z_{G_k}(\overline{\rho}(\gamma))^\circ$ equal to the split maximal torus T_k, and for which there is a unique root $\alpha \in \Phi(G, T)$ satisfying $\alpha(\overline{\rho}(\gamma)) = \overline{\chi}(\gamma) \neq 1$. (If $\dim T = 1$, we furthermore require that $\overline{\chi}(\gamma)^3 \neq 1$. This is used only in the proof of Lemma 2.4.2.3 in cases with G of rank 1.)

Remark 2.3.1.2. In (iii), note that $\alpha(\overline{\rho}(\gamma))$ makes sense because $\overline{\rho}(\gamma) \in T(k)$, as any semisimple element $g \in G(k)$ satisfies $g \in Z_{G_k}(g)^\circ$.

Remark 2.3.1.3. Note that if we extend k the representation remains big: extending the field does not change the cohomological vanishing results in the second condition, and the first and third are unaffected by the extension.

Our goal is to prove a sufficient, easy-to-check condition for a representation to be big. A version of this argument goes back to [Ram99], and is similar to the independently-worked-out arguments in [Pat15, §6]. The main assumptions which will imply bigness are that the image of $\overline{\rho}$ contains $G'(k)$ and the representation $\mathrm{ad}^0(\overline{\rho})$ is absolutely irreducible. For this to work, we will need the following additional assumptions (easily checkable in practice):

(L1) p is a very good prime for G, $q \neq 2, 3, 4, 5, 9$, and $\mathbf{Q}(\zeta_p) \cap K = \mathbf{Q}$ (so $[K(\zeta_p) : K] = p - 1$);

(L2) $K(\mathrm{ad}^0(\overline{\rho}))$ does not contain ζ_p (in particular, the cyclic group $\overline{\chi}(\Gamma_{K(\mathrm{ad}^0(\overline{\rho}))}) \subset \mathbf{F}_p^\times$ has order $d > 1$);

(L3) there exists a regular semisimple element $g \in [G'(k), G'(k)] \subset G'(k)$ with $Z_{G_k}(g) = T_k$, and a non-trivial element $x \in \overline{\chi}(\Gamma_{K(\mathrm{ad}^0(\overline{\rho}))}) \subset \mathbf{F}_p^\times$ such that there is a unique root $\alpha \in \Phi(G, T)$ for which $\alpha(g) = x$. (If $\dim T = 1$, we furthermore require that $\overline{\chi}(g)^3 \neq 1$.)

We will show these assumptions are automatic for large enough primes when the image of $\overline{\rho}$ contains $G'(k)$, and hold for many smaller primes as well. First, we will show these assumption imply bigness.

Proposition 2.3.1.4. *Under assumptions (L1), (L2), and (L3), $\overline{\rho}$ is a big representation provided the image of $\overline{\rho}$ contains $G'(k)$ and $\mathrm{ad}^0(\overline{\rho})$ is an absolutely irreducible representation of Γ_K.*

Remark 2.3.1.5. Patrikis instead uses the condition $G'(k') \subset \overline{\rho}(\Gamma_K) \subset Z(k)G(k')$ for some subfield $k' \subset k$. His proof uses the same method.

We now assume (L1), (L2), and (L3), that $G'(k) \subset \overline{\rho}(\Gamma_K)$, and that $\mathrm{ad}^0(\overline{\rho})$ is an absolutely irreducible representation of Γ_K.

We will prove condition (ii) of Definition 2.3.1.1 holds when $G'(k) \subset \overline{\rho}(\Gamma_K)$. According to Proposition 2.1.2.4, $H^1(G'(k), \mathfrak{g}'_k) = 0$. We will now relate this to Galois cohomology using the inflation-restriction sequence. To do so, we need several lemmas.

Lemma 2.3.1.6. *We have naturally $\overline{\rho}(\Gamma_K)/(Z_G(k) \cap \overline{\rho}(\Gamma_K)) \xrightarrow{\sim} \mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/K)$.*

Proof. Lemma 2.1.1.6 gives a decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}_{\mathfrak{g}}$. The kernel of $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ is Z_G, and the action is trivial on $\mathfrak{z}_{\mathfrak{g}}$. Thus the kernel of $\mathrm{Ad}_G \circ \overline{\rho}$ equals $\overline{\rho}^{-1}(Z_G(k))$, and is also the kernel of the action of Γ_K on $\mathrm{ad}^0(\overline{\rho})$. The Galois group $\mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/K)$ is the quotient. ☐

Lemma 2.3.1.7. *The index of $G'(k)$ in $G(k)$ is prime to p.*

Proof. We have an exact sequence

$$1 \to G'(k) \to G(k) \to (G/G')(k)$$

But G/G' is a split k-torus, so the number of its k-points is relatively prime to p. ☐

Lemma 2.3.1.8. *The index of $\Gamma_{K(\mathrm{ad}^0(\overline{\rho})^*)}$ in $\Gamma_{K(\mathrm{ad}^0(\overline{\rho}))}\Gamma_{K(\mathrm{ad}^0(\overline{\rho})^*)}$ is prime to p.*

Proof. Elementary group theory shows that the index is equal to the index of $\Gamma_{K(\mathrm{ad}^0(\overline{\rho}))} \cap \Gamma_{K(\mathrm{ad}^0(\overline{\rho})^*)}$ in $\Gamma_{K(\mathrm{ad}^0(\overline{\rho}))}$. But this subgroup is the kernel of the cyclotomic character $\overline{\chi} : \Gamma_{K(\mathrm{ad}^0(\overline{\rho}))} \to k^\times$. Thus the index is relatively prime to p. ☐

Now consider the inflation-restriction sequence

$$0 \to H^1(\overline{\rho}(\Gamma_K)/G'(k), \mathrm{ad}^0(\overline{\rho})^{G'(k)}) \to H^1(\overline{\rho}(\Gamma_K), \mathrm{ad}^0(\overline{\rho})) \to H^1(G'(k), \mathrm{ad}^0(\overline{\rho}))$$

We know the third term is 0 by Proposition 2.1.2.4. The first term is zero as the index of $G'(k)$ in $\overline{\rho}(\Gamma_K)$ is prime to p because of Lemma 2.3.1.7 and $G'(k) \subset \overline{\rho}(\Gamma_K) \subset G(k)$. This implies that $H^1(\overline{\rho}(\Gamma_K), \mathrm{ad}^0(\overline{\rho})) = 0$. Using Lemma 2.3.1.6, another application of the inflation-restriction sequence gives

$$0 \to H^1(\mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/K), \mathrm{ad}^0(\overline{\rho})^{Z_G(k) \cap \overline{\rho}(\Gamma_K)}) \to H^1(\overline{\rho}(\Gamma_K), \mathrm{ad}^0(\overline{\rho}))$$

But the right term is zero and $Z_G(k)$ acts trivially on $\mathrm{ad}^0(\overline{\rho})$, so we conclude that

$$H^1(\mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/K), \mathrm{ad}^0(\overline{\rho})) = 0.$$

We also claim that $H^1(\mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho})^*)/K), \mathrm{ad}^0(\overline{\rho})^*) = 0$. To prove this, we will use the subgroup

$$\Gamma' := \Gamma_{K(\mathrm{ad}^0(\overline{\rho}))} \cdot \Gamma_{K(\mathrm{ad}^0(\overline{\rho})^*)}/\Gamma_{K(\mathrm{ad}^0(\overline{\rho})^*)} \subset \Gamma_K/\Gamma_{K(\mathrm{ad}^0(\overline{\rho})^*)} =: \Gamma.$$

The inflation-restriction sequence begins

$$0 \to H^1(\Gamma/\Gamma', \left(\mathrm{ad}^0(\overline{\rho})^*\right)^{\Gamma'}) \to H^1(\Gamma_K/\Gamma_{K(\mathrm{ad}^0(\overline{\rho})^*)}, \mathrm{ad}^0(\overline{\rho})^*) \to H^1(\Gamma', \mathrm{ad}^0(\overline{\rho})^*).$$

The action of $\Gamma_{K(\mathrm{ad}^0(\overline{\rho}))}$ on $\mathrm{ad}^0(\overline{\rho})^*$ is via the cyclotomic character: by (L2) this character is non-trivial, so $\left(\mathrm{ad}^0(\overline{\rho})^*\right)^{\Gamma'} = 0$. Thus the first term is 0. The third term is zero because Γ' is a finite group of order prime to p (Lemma 2.3.1.8). This gives the desired vanishing, establishing (ii) in Definition 2.3.1.1.

We finally turn to producing $\gamma \in \Gamma_K$ as in Definition 2.3.1.1(iii). Let $L = K(\mathrm{ad}^0(\overline{\rho})) \cap K(\zeta_p)$, and $K(\mathrm{ad}^0(\overline{\rho}), \zeta_p)$ be the compositum of $K(\mathrm{ad}^0(\overline{\rho}))$ and $K(\zeta_p)$. By Galois theory, there is a surjection

$$\Gamma_L \twoheadrightarrow \mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}), \zeta_p)/L) \simeq \mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/L) \times \mathrm{Gal}(K(\zeta_p)/L). \tag{2.3.1.1}$$

Furthermore, $\mathrm{Gal}(K(\zeta_p)/L) \simeq \mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}), \zeta_p)/K(\mathrm{ad}^0(\overline{\rho}))) = \overline{\chi}(\Gamma_{K(\mathrm{ad}^0(\overline{\rho}))}) \subset \mathbf{F}_p^\times$. We also note that $\mathrm{Gal}(L/K)$ is an abelian quotient of $\mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/K) \simeq \overline{\rho}(\Gamma_K)/\left(Z_G(k) \cap \overline{\rho}(\Gamma_K)\right)$. This means that

$$[\overline{\rho}(\Gamma_K)/\left(Z_G(k) \cap \overline{\rho}(\Gamma_K)\right), \overline{\rho}(\Gamma_K)/\left(Z_G(k) \cap \overline{\rho}(\Gamma_K)\right)] \subset \mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/L)$$

and as $G'(k) \subset \overline{\rho}(\Gamma_K)$ we conclude

$$[G'(k)/Z_{G'}(k), G'(k)/Z_{G'}(k)] \subset \mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/L).$$

Given $g \in [G'(k), G'(k)]$ and $x \in \overline{\chi}(\Gamma_{K(\mathrm{ad}^0(\overline{\rho}))})$ as in (L3), using (2.3.1.1) we see there exists $\gamma \in \Gamma_L \subset \Gamma_K$ such that $\overline{\chi}(\gamma) = x$ (so $\overline{\chi}(\gamma)^3 \neq 1$ when $\dim T = 1$) and $\overline{\rho}(\gamma) = gz$ for some $z \in Z_G(k) \cap \overline{\rho}(\Gamma_K)$. We are given that there is a unique root $\alpha \in \Phi(G, T)$ such that $\alpha(g) = x$. We see that $Z_{G_k}(\overline{\rho}(\gamma)) = Z_{G_k}(g) = T_k$ and $\alpha(\overline{\rho}(\gamma)) = \alpha(g) = x$ as desired. This completes the proof of Proposition 2.3.1.4. ☐

2.3.2 Checking the Assumptions

Now assume that $G'(k) \subset \overline{\rho}(\Gamma_K)$ and (L1) holds. We wish to understand when $\mathrm{ad}^0(\overline{\rho})$ is an absolutely irreducible, and when (L2) and (L3) hold, in terms of the root datum $\Phi(G, T)$. Let $\pi : \widetilde{G'} \to G'$ be the simply connected central cover.

Lemma 2.3.2.1. *If $\Phi(G, T)$ is irreducible, $\mathrm{ad}^0(\overline{\rho})$ is an absolutely irreducible Γ_K-representation.*

Proof. By assumption $\widetilde{G'}$ is absolutely simple. As $q > 3$, Lemma 2.1.2.1 implies that the adjoint representation of $\widetilde{G'}(k)$ on $(\mathrm{Lie}\, G')_k$ is absolutely irreducible as p is very good for G. Again using that p is very good, Lemma 2.1.1.5 implies $(\mathrm{Lie}\, \widetilde{G'})_k \simeq \mathfrak{g}'_k$. As $\pi(\widetilde{G'}(k)) \subset G'(k) \subset \overline{\rho}(\Gamma_K)$, we conclude that $\mathrm{ad}^0(\overline{\rho})$ is an absolutely irreducible Γ_K-representation. ☐

Remark 2.3.2.2. Patrikis identifies some more precise conditions than absolute irreducibility that work in Ramakrishna's method [Pat15, (1), (5), (6bc) of §5]. He then checks that they hold when G is simple. We have chosen to use absolute irreducibility for convenience.

The kernel of $f : Z_G \times \widetilde{G'} \to G$, being a subgroup of $Z_{\widetilde{G'}}$, has order prime to p as p is a very good prime for G.

Lemma 2.3.2.3. *If the image of $\overline{\rho}$ contains $G'(k)$ then $\#\,\mathrm{Gal}(K(\zeta_p) \cap K(\mathrm{ad}^0(\overline{\rho}))/K)$ divides $\# H^1(k, \ker f)$. In particular, if $H^1(k, \ker f)$ does not contain any elements of order $p-1$ then $\zeta_p \notin K(\mathrm{ad}^0(\overline{\rho}))$.*

Proof. The final assertion follows from the rest since $[K(\zeta_p) : K] = p - 1$ by (L1). We will show that the abelianization of $\mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/K)$ is a subquotient of $G(k)/Z_G(k){\cdot}\pi(\widetilde{G'}(k))$, and that $G(k)/Z_G(k){\cdot}\pi(\widetilde{G'}(k))$ has order dividing $\# H^1(k, \ker f)$.

As $G'(k) \subset \overline{\rho}(\Gamma_K)$, observe that

$$G'(k)/(G'(k) \cap Z_G(k)) \subset \overline{\rho}(\Gamma_K)/(\overline{\rho}(\Gamma_K) \cap Z_G(k)) \subset G(k)/Z_G(k).$$

By Lemma 2.3.1.6,

$$\overline{\rho}(\Gamma_K)/(\overline{\rho}(\Gamma_K) \cap Z_G(k)) \simeq \mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/K).$$

Thus we see that $\mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/K)^{\mathrm{ab}}$ is a sub-quotient of $G(k)/Z_G(k)[G'(k), G'(k)]$.

Now consider the map

$$f : Z_G \times \widetilde{G'} \to Z_G \times G' \to G$$

By Lemma 2.1.2.2, $[G'(k), G'(k)] = \pi(\widetilde{G'}(k))$ as $q > 3$. From the long exact sequence

$$Z(k) \times \widetilde{G'}(k) \to G(k) \to H^1(k, \ker f)$$

we conclude that $\mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/K)^{\mathrm{ab}}$ is a subquotient of $H^1(k, \ker f)$. ☐

In particular, the group $H^1(k, \ker f)$ is finite and killed by $\#\ker(f)$, so $p-1 > \#\ker(f)$ is sufficient for (L2). If G'_k has root system Φ, we know that $\#\ker(f)|\#\pi_1(\Phi)$, so $p > \#\pi_1(\Phi)$ suffices. Thus (L2) holds for sufficiently large p for a group G with fixed root datum. In any particular instance, this is easy to make explicit.

Example 2.3.2.4. Take $G = \mathrm{GSp}_{2n}$. Then $G' = \mathrm{Sp}_{2n}$ is simply connected, and $\ker f = Z_{\mathrm{Sp}_{2n}} = \mu_2$. Thus any prime bigger than 3 is fine.

Finally, we give a sufficient condition for (L3) to hold. It is far from necessary.

Let g be a generator of $\mathbf{F}_p^\times$, d the order of $\overline{\chi}(\Gamma_{K(\mathrm{ad}^0(\overline{\rho}))})$, and $e = \frac{p-1}{d} = [K(\zeta_p) \cap K(\mathrm{ad}^0(\overline{\rho})) : K]$. Lemma 2.3.2.3 shows that for fixed root system, e divides $\#H^1(k, \ker f)$ which is bounded independent of p. To be precise, writing $\ker(f) = \prod \mu_{n_i}$, we see

$$\#H^1(k, \ker f) = \prod_i \gcd(n_i, q-1) \mid \prod n_i = \#\ker f \mid \#\pi_1(\Phi).$$

Thus for large enough p we have that $d > 4(h-1)$, where h is the Coxeter number. Recall that upon picking a basis Δ of the root system, the *height* $\mathrm{ht}(\beta)$ of a root β is the the sum of the coefficients when the root is written in terms of Δ. The Coxeter number h is 1 more than the height of the highest root.

Lemma 2.3.2.5. *Suppose* $\dim T \neq 1$. *Condition (L3) holds when* $d > 4(h-1)$.

Proof. We will work with the simply connected central cover $\pi : \widetilde{G'} \to G'$, with $\widetilde{T'}$ the split maximal torus preimage of $T \cap G' \subset G'$. Consider the root system $\Phi = \Phi(\widetilde{G'}, \widetilde{T'})$ with chosen basis of positive roots Δ, and the cocharacter $\delta \in X_*(\widetilde{T'})$ which is the sum of the positive coroots of Φ. Take β to be an element of order d in $\overline{\chi}(\Gamma_{K(\mathrm{ad}^0(\overline{\rho}))}) \subset \mathbf{F}_p^\times \subset k^\times$, and let $s = \delta(\beta) \in \widetilde{G'}(k)$. Recall that for $\alpha_i \in \Delta$, $\langle \delta, \alpha_i \rangle = 2$. Thus we calculate that for any root α,

$$\alpha(s) = \beta^{2\,\mathrm{ht}(\alpha)}.$$

In particular, if α is the highest root then we obtain $\beta^{2(h-1)}$. For any other root α', $0 < 2\,\mathrm{ht}(\alpha') < 2(h-1)$ or $-2(h-1) \leq 2\,\mathrm{ht}(\alpha') < 0$ and hence as $d > 4(h-1)$

$$2\,\mathrm{ht}(\alpha) \not\equiv 2\,\mathrm{ht}(\alpha') \pmod{d} \quad \text{and} \quad 2\,\mathrm{ht}(\alpha') \not\equiv 0 \pmod{d}.$$

Thus $\alpha'(s) \neq \alpha(s)$ and $\alpha'(s) \neq 1$. Therefore we conclude that that s is regular semisimple and α is the unique root on which the adjoint action on $(\mathrm{Lie}\,\widetilde{G'})_k$ is given by $\beta^{2(h-1)} \in \overline{\chi}(\Gamma_{K(\mathrm{ad}^0(\overline{\rho}))})$.

As p is very good, $d\pi_k : (\mathrm{Lie}\,\widetilde{G'})_k \to (\mathrm{Lie}\,G')_k$ is an isomorphism. Thus $\pi(s) \in \pi(\widetilde{G'}(k)) = [G'(k), G'(k)]$ (using Lemma 2.1.2.2) has the required properties. ☐

If $\dim T = 1$, a minor modification shows that (L3) holds as long as $d > 6$. In particular, for fixed root datum and sufficiently large primes p, (L3) holds.

Remark 2.3.2.6. The above condition is far from sharp. There is no reason that the highest root must be the unique root α in (L3) where the adjoint action is given by the cyclotomic character.

Example 2.3.2.7. Let $G = \mathrm{GL}_n$. Then the Coxeter number is n, and $Z_{G'} = \mu_n$. Let $e = (n, p-1)$, so $H^1(k, \mu_n) = \mathbf{Z}/e\mathbf{Z}$. Assumptions (L1)-(L3) hold provided $p > \max(5, 4(n-1)e)$.

For $G = \mathrm{Sp}_{2n}$, the Coxeter number is $2n$, and $Z_{G'} = \mu_2$. Then $H^1(k, \mu_2) = \mathbf{Z}/2\mathbf{Z}$ and the assumptions hold when $p > \max(5, 8(2n-1))$. There is a similar bound for orthogonal groups.

2.4 Ramakrishna's Deformation Condition

Let $\mathcal{O}$ be the ring of integers in a p-adic field with residue field k. Consider a split connected reductive group scheme G over $\mathcal{O}$. Let K be a number field and $\overline{\rho} : \Gamma_K \to G(k)$ be a big representation. Recall that $\chi : \Gamma_K \to \mathbf{Z}_p^\times$ denotes the cyclotomic character.

In this section, we will generalize a deformation condition that Ramakrishna introduces for GL_2, allowing controlled ramification at certain unramified places. Allowing ramification subject to this condition is a crucial tool to reduce the size of the dual Selmer group. A similar generalization is carried out in [Pat15, §4.2].

2.4.1 Constructing the Deformation Condition

As $\overline{\rho}$ is big, there is a $\gamma \in \Gamma_K$ such that $\overline{\rho}(\gamma)$ is regular semisimple. The identity component of $Z_{G_k}(\overline{\rho}(\gamma))$ is T_k, where T was a specified split maximal torus of G. By hypothesis, T is split and there is a unique root $\alpha \in \Phi(G,T)$ such that $\alpha(\overline{\rho}(\gamma)) = \overline{\chi}(\gamma)$. We start by assuming:

(A1) there is a place v of K lying over a rational prime ℓ such that $\overline{\rho}$ is unramified at v and $\overline{\rho}(\mathrm{Frob}_v)$ is regular semisimple element. The identity component of $Z_{G_k}(\overline{\rho}(\mathrm{Frob}_v))$ is T_k, and there is a unique root $\alpha \in \Phi(G,T)$ such that $\alpha(\overline{\rho}(\mathrm{Frob}_v)) = \overline{\chi}(\mathrm{Frob}_v) = \overline{\chi}(\gamma) \neq 1$.

Under this assumption, we will define a deformation condition consisting of certain tamely ramified lifts. Let K_v^{t} be the maximal tamely ramified extension of K_v, K_v^{nr} be the maximal unramified extension, and $\Gamma_v^{\mathrm{t}} = \mathrm{Gal}(K_v^{\mathrm{t}}/K_v)$. There is a split exact sequence

$$1 \to \mathrm{Gal}(K_v^{\mathrm{t}}/K_v^{\mathrm{nr}}) \to \Gamma_v^{\mathrm{t}} \to \mathrm{Gal}(K_v^{\mathrm{nr}}/K_v) \to 1.$$

Recall that $\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v)$ is topologically generated by Frob_v, while $\mathrm{Gal}(K_v^{\mathrm{t}}/K_v^{\mathrm{nr}}) \simeq \prod_{p' \neq \ell} \mathbf{Z}_{p'}(1)$. The action of $\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v)$ on $\mathrm{Gal}(K_v^{\mathrm{t}}/K_v^{\mathrm{nr}})$ is via the prime-to-ℓ cyclotomic character. Concretely, the action of Frob_v is given by multiplication by an integer $q = \ell^{f(K_v/\mathbf{Q}_\ell)}$. For a fixed splitting, we obtain a semidirect product decomposition

$$\Gamma_v^{\mathrm{t}} \simeq \mathrm{Gal}(K_v^{\mathrm{nr}}/K_v) \ltimes \mathrm{Gal}(K_v^{\mathrm{t}}/K_v^{\mathrm{nr}}). \tag{2.4.1.1}$$

Fix a split maximal $\mathcal{O}$-torus T of G reducing to T_k: we identify $\alpha \in \Phi(G,T) = \Phi(G_k, T_k)$ and form the usual smooth closed $\mathcal{O}$-subgroups $U_\alpha \subset G$ (root group for α) and $H_\alpha = T \ltimes U_\alpha \subset G$ associated to α and to $\{0, \alpha\}$ (see Theorem 5.1.13 and Proposition 5.1.16 of [Con14]).

Definition 2.4.1.1. We assume (A1). For a coefficient $\mathcal{O}$-algebra A, consider a lift $\rho : \Gamma_v^{\mathrm{t}} \to G(A)$. The lift ρ satisfies *Ramakrishna's condition relative to* T provided that $\rho(\mathrm{Frob}_v) \in T(A)$, $\alpha(\rho(\mathrm{Frob}_v)) = \chi(\mathrm{Frob}_v)$, and $\rho(\mathrm{Gal}(K_v^{\mathrm{t}}/K_v^{\mathrm{nr}})) \subset U_\alpha(A) \subset G(A)$.

Define *Ramakrishna's deformation condition* $\mathcal{D}_v^{\mathrm{ram}}(A)$ to be lifts which are $\widehat{G}(A)$-conjugate to one which satisfies Ramakrishna's condition relative to T. Denote the tangent space of the deformation functor by L_v^{ram}.

This condition generalizes the condition for GL_2 introduced by Ramakrishna [Ram99, §3]. Note that the condition can be rephrased as $\rho \in \mathcal{D}_v^{\mathrm{ram}}(A)$ if and only if there exists a choice of split $\mathcal{O}$-torus T lifting T_k such that ρ satisfies Ramakrishna's condition relative to T since $\widehat{G}(\mathcal{O})$ acts transitively on the set of such T by [Con14, Theorem 3.2.6].

Example 2.4.1.2. Let $G = \mathrm{GL}_n$. We may assume that the residual representation, which is unramified, sends Frob_v to a diagonal element. Ramakrishna's deformation condition consists of lifts such that after conjugation $\rho(\mathrm{Frob}_v)$ is diagonal and the the image of $\mathrm{Gal}(K_v^{\mathrm{t}}/K_v^{\mathrm{nr}})$ consists of elements of the shape

$$\begin{pmatrix} 1 & * & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Conjugation by Frob_v is multiplication by $q = \chi(\mathrm{Frob}_v)$ on $\mathrm{Gal}(K_v^{\mathrm{t}}/K_v^{\mathrm{nr}})$. On the other hand, the action of $\rho(\mathrm{Frob}_v)$ on U_α is given by $\alpha(\rho(\mathrm{Frob}_v))$. Thus the assumption that $\alpha(\overline{\rho}(\mathrm{Frob}_v)) = \overline{\chi}(\mathrm{Frob}_v)$ is crucial to allow the component $*$ to be non-zero and produce lifts where $\rho(\mathrm{Gal}(K_v^{\mathrm{t}}/K_v^{\mathrm{nr}})) \neq 1$.

Lemma 2.4.1.3. *$\mathcal{D}_v^{\mathrm{ram}}$ is a deformation condition.*

Proof. $\mathcal{D}_v^{\mathrm{ram}}(A)$ is certainly closed under strict equivalence. Next, we will check it is subfunctor of the deformation functor at $\overline{\rho}$. Let $f : A_1 \to A_0$ be a morphism of coefficient $\mathcal{O}$-algebras, and choose $\rho_1 \in \mathcal{D}_v^{\mathrm{ram}}(A_1)$. This means there is a split maximal torus T defined over $\mathcal{O}$ such that $\rho(\mathrm{Frob}_v) \in T(A_1)$, $\alpha(\rho(\mathrm{Frob}_v)) = \chi(\mathrm{Frob}_v)$, and $\rho(\mathrm{Gal}(K_v^{\mathrm{t}}/K_v^{\mathrm{nr}})) \subset U_\alpha(A_1)$. The same torus shows that $f(\rho) \in \mathcal{D}_v^{\mathrm{ram}}(A_0)$.

It remains to check condition (2) of Definition 2.2.2.7. Let T be a fixed split $\mathcal{O}$-torus lifting T_k, and let A_1, A_2, and A_0 be coefficient $\mathcal{O}$-algebras with morphisms $f_1 : A_1 \to A_0$ and $f_2 : A_2 \to A_0$. We can reduce to the case that f_2 is small. It suffices to show that given deformations $\rho_i \in \mathcal{D}^{\square}_{\overline{\rho}}(A_i)$ such that $\mathcal{D}^{\square}_{\overline{\rho}}(f_1)(\rho_1) = \mathcal{D}^{\square}_{\overline{\rho}}(f_2)(\rho_2) = \rho_0$, if ρ_1 and ρ_2 satisfy Ramakrishna's lifting condition then so does $\rho_1 \times_{\rho_0} \rho_2$. This means that there are $g_i \in \widehat{G}(A_i)$ such that $\rho_i^{g_i}$ satisfies Ramakrishna's condition relative to T. In particular, the push-forwards of $\rho_1^{g_1}$ and $\rho_2^{g_2}$ to A_0 both satisfy Ramakrishna's condition relative to T, and letting $h = g_1 g_2^{-1}$ both $\rho_0^{g_2}$ and $\rho_0^{hg_2} = (\rho_0^{g_2})^h$ satisfy Ramakrishna's condition relative to T. We will show that this implies $h \in \widehat{H}_\alpha(A_0)$. Using the smoothness of this group, we can lift to an element $h' \in \widehat{H}_\alpha(A_2)$. But then $(g_1, hg_2) \in \widehat{G}(A_1) \times_{\widehat{G}(A_0)} \widehat{G}(A_0) = \widehat{G}(A_1 \times_{A_0} A_2)$, and this element conjugates $\rho_1 \times_{\rho_0} \rho_2$ to satisfy Ramakrishna's condition relative to T.

It remains to prove the following statement: if $\rho : \Gamma_v \to G(R)$ satisfies Ramakrishna's condition relative to T, then

$$C_\rho(R) := \{g \in \widehat{G}(R) : \rho^g(\Gamma_v) \subset H_\alpha(R)\} \subset \widehat{H}_\alpha(R).$$

If $R = k$, this is trivial. In general, argue by induction on the length of R. It suffices to consider a small morphism $R \to R/I$ and assume the statement for R/I. For $g \in C_\rho(R)$, induction shows that $\overline{g} \in \widehat{H}_\alpha(R/I)$ hence it suffices to check that $\exp(x) \in \widehat{H}_\alpha(R)$ for all $x \in \mathfrak{g} \otimes_k I$ for which $\rho^{\exp(x)}(\Gamma_v) \subset H_\alpha(R)$. Rewriting, we see that

$$\exp(x)\rho(\mathrm{Frob}_v)\exp(-x) \in H_\alpha(R).$$

Multiplying on the right by $\rho(\mathrm{Frob}_v)^{-1} \in H_\alpha(R)$, we conclude that

$$x - \mathrm{Ad}_G(\overline{\rho}(\mathrm{Frob}_v))x \in (\mathrm{Lie}\, H_\alpha)_k \otimes_k I.$$

As $\overline{\rho}(\mathrm{Frob}_v)$ is regular semisimple, it acts non-trivially on $\mathfrak{g}_\beta$ for all $\beta \in \Phi(G, T)$, forcing $x \in (\mathrm{Lie}\, H_\alpha)_k \otimes_k I$ as desired. □

Now we will describe the tangent space to the deformation condition. Fix a choice of $\mathcal{O}$-torus T lifting T_k, and let $\mathfrak{g} := \mathrm{Lie}\, G_k$ and $\mathfrak{t} = \mathrm{Lie}\, T_k$. We have a decomposition

$$\mathrm{ad}(\overline{\rho}) = \mathfrak{t} \oplus \bigoplus_{\beta \in \Phi(G,T)} \mathfrak{g}_\beta. \tag{2.4.1.2}$$

Because $\mathrm{ad}(\overline{\rho})$ is unramified at v, the Galois action is completely determined by the action of Frob_v, which acts on $\mathfrak{g}_\beta$ as $\beta(\overline{\rho}(\mathrm{Frob}_v))$. This will allow us to compute Galois cohomology at v easily:

Lemma 2.4.1.4. *We have* $\dim H^0(\Gamma_v, \mathrm{ad}(\overline{\rho})) = \dim \mathfrak{t}$ *and* $\dim H^1(\Gamma_v, \mathrm{ad}(\overline{\rho})) = \dim \mathfrak{t} + 1$. *Furthermore,*

$$H^1(\Gamma_v, \mathrm{ad}(\overline{\rho})) = H^1(\Gamma_v, \mathfrak{t}) \oplus H^1(\Gamma_v, \mathfrak{g}_\alpha).$$

Proof. It is straightforward to see that $\dim H^0(\Gamma_v, \mathrm{ad}(\overline{\rho})) = \dim H^0(\Gamma_v, \mathfrak{t}) = \dim \mathfrak{t}$ using (2.4.1.2): since $\overline{\rho}(\mathrm{Frob}_v)$ is regular semisimple, its space of fixed vectors in $\mathfrak{g}$ is $\mathfrak{t}$. We claim that $\dim_k H^2(\Gamma_v, \mathrm{ad}(\overline{\rho})) = \dim H^2(\Gamma_v, \mathfrak{g}_\alpha) = 1$. This follows from Fact 2.2.1.1 as the only piece of the decomposition on which Frob_v acts as the cyclotomic character is $\mathfrak{g}_\alpha$. Then Fact 2.2.1.2 implies that $\dim_k H^1(\Gamma_v, \mathrm{ad}(\overline{\rho})) = \dim \mathfrak{t} + 1$ and gives the decomposition of $H^1(\Gamma_v, \mathrm{ad}(\overline{\rho}))$. □

There are also some obvious deformations to $k[\epsilon]/\epsilon^2$ of the form $\rho_i = \exp(\epsilon f_i)\overline{\rho}$ where $f_i \in Z^1(\Gamma_v, \mathrm{ad}(\overline{\rho}))$ is defined as follows. Choose any non-zero homomorphism $f_0' : \mathrm{Gal}(K_v^{\mathrm{t}}/K_v^{\mathrm{nr}}) \to \mathbf{Z}/p\mathbf{Z} \to \mathfrak{g}_\alpha$, and define $f_0 : \Gamma_v \to \mathfrak{g}_\alpha$ via the composition

$$\Gamma_v \to \Gamma_v^{\mathrm{t}} \xrightarrow{\pi} \mathrm{Gal}(K_v^{\mathrm{t}}/K_v^{\mathrm{nr}}) \xrightarrow{f_0'} \mathfrak{g}_\alpha$$

where π is the projection determined by the splitting (2.4.1.1). It is straightforward to check this is a cocycle using that $\alpha(\overline{\rho}(\mathrm{Frob}_v)) = \overline{\chi}(\mathrm{Frob}_v)$. Now choose a basis $t_1, \ldots, t_{\dim \mathfrak{t}}$ for $\mathfrak{t}$ such that $t_1, \ldots, t_{\dim \mathfrak{t} - 1} \in \mathfrak{t}_\alpha$, where $\mathfrak{t}_\alpha$ is the Lie algebra of the codimension-1 subtorus $\ker(\alpha)^0_{\mathrm{red}}$. Define an unramified homomorphism $f_i : \Gamma_v \to \Gamma_v^{\mathrm{nr}} \to \mathfrak{t}$ by sending Frob_v to t_i: it is clearly a cocycle since $\overline{\rho}(\mathrm{Frob}_v) \in T(k)$.

Lemma 2.4.1.5. *We have* $\dim L_v^{\mathrm{ram}} = \dim \mathfrak{t} = \dim H^0(\Gamma_v, \mathrm{ad}(\overline{\rho}))$. *The cocycles* $f_0, f_1, \ldots, f_{\dim \mathfrak{t} - 1}$ *form a basis for* $L_v^{\mathrm{ram}} \subset H^1(\Gamma_v, \mathrm{ad}(\overline{\rho}))$.

Proof. First, we check that the cocycles $f_0, f_1, \ldots, f_{\dim \mathfrak{t}}$ form a basis for $H^1(\Gamma_v, \mathrm{ad}(\overline{\rho}))$: by Lemma 2.4.1.4, it suffices to check they are linearly independent (modulo coboundaries). Note that for any 1-coboundary

$$c_x : g \mapsto \mathrm{ad}(\overline{\rho})(g)(x) - x$$

with $x \in \mathfrak{g}$, $c(\mathrm{Frob}_v)$ has vanishing $\mathfrak{t}$-component. This immediately gives that $f_1, \ldots f_{\dim \mathfrak{t}}$ are linearly independent modulo coboundaries. Furthermore, by inspection f_0 is a nonzero element of $H^1(\Gamma_v, \mathfrak{g}_\alpha)$. In light of the decomposition from Lemma 2.4.1.4, we obtain linear independence of $f_0, f_1, \ldots, f_{\dim \mathfrak{t}}$.

Finally, we consider which of the lifts $\rho_i = \exp(f_i \epsilon)\overline{\rho}$ satisfy the deformation condition $\mathcal{D}_v^{\mathrm{ram}}$. The requirement is that $\alpha(\rho_i(\mathrm{Frob}_v)) = \overline{\chi}(\mathrm{Frob}_v) \in k[\epsilon]/(\epsilon^2)$. Now $\mathrm{ad}(\overline{\rho})(\mathrm{Frob}_v)$ acts on $\mathfrak{g}_\alpha$ via multiplication by $\overline{\chi}(\mathrm{Frob}_v)$ by (A1). Furthermore, $\mathrm{ad}(\exp(f_i(\mathrm{Frob}_v)\epsilon)) = [f_i(\mathrm{Frob}_v), -]$ on $\mathfrak{g}$, and on $\mathfrak{g}_\alpha$ the action is multiplication by $\mathrm{Lie}(\alpha)(f_i(\mathrm{Frob}_v))$. So ρ_i satisfies $\mathcal{D}_v^{\mathrm{ram}}$ if and only if $f_i(\mathrm{Frob}_v) \in \mathfrak{t}_\alpha$. We see that $f_0, f_1, \ldots, f_{\dim \mathfrak{t}-1}$ satisfy this requirement but $f_{\dim \mathfrak{t}}$ does not. □

Lemma 2.4.1.6. *$\mathcal{D}_v^{\mathrm{ram}}$ is liftable.*

Proof. To see it is liftable, consider trying to lift $\rho : \Gamma_v^{\mathrm{t}} \to G(A/I)$ to A, where $A \to A/I$ is a small extension. There is a split maximal torus T of G over $\mathcal{O}$ such that ρ factors through $T(A/I) \ltimes U_\alpha(A/I)$. Write

$$\rho(g) = t(g) \ltimes u_\alpha(x(g))$$

where $t \in \mathrm{Hom}(\Gamma_v^{\mathrm{t}}, T(A/I))$, $u_\alpha : \mathbf{G}_a \xrightarrow{\sim} U_\alpha$ and $x \in Z^1(\Gamma_v^t, \chi_{A/I})$ because $\alpha(\mathrm{Frob}_v)) = \chi_{A/I}(\mathrm{Frob}_v)$. (Note that the Galois action on $U_\alpha(R/I) = \mathbf{G}_a(R/I) = R/I$ is given by multiplication against the cyclotomic character $\chi_{A/I}$.) To lift ρ, it suffices to continuously lift t and x as a homomorphism and tame 1-cocycle respectively, as then the combined lift will determine a lift of ρ.

We can easily lift t: it is unramified, so just lift an image of Frob_v using the smoothness of the torus. To lift x, we claim that $H^1(\Gamma_v^{\mathrm{t}}, \chi_A) \to H^1(\Gamma_v^{\mathrm{t}}, \chi_{A/I})$ is surjective. To check this, it suffices to check that the next piece of the long exact sequence, $H^2(\Gamma_v^{\mathrm{t}}, \chi_I) \to H^2(\Gamma_v^{\mathrm{t}}, \chi_A)$, is injective. Since $v \nmid p$,

$$H^2(\Gamma_v^{\mathrm{t}}, \chi_I) = H^2(\Gamma_v, \chi_I) \quad \text{and} \quad H^2(\Gamma_v^{\mathrm{t}}, \chi_A) = H^2(\Gamma_v, \chi_A).$$

Using local duality, the claim about injectivity reduces to the evident surjectivity of $\mathrm{Hom}(A, \mathbf{Q}/\mathbf{Z}) \to \mathrm{Hom}(I, \mathbf{Q}/\mathbf{Z})$. Thus there is a cocycle $x' \in Z^1(\Gamma_v^{\mathrm{t}}, \chi_A)$ which reduces to x up to a (tame) 1-coboundary. But a tame coboundary valued in the unramified χ_A obviously lifts. Therefore we can lift both t and x, hence we can produce a homomorphism $\rho' : \Gamma_v^{\mathrm{t}} \to T(A) \ltimes U_\alpha(A)$ lifting ρ. □

Putting this all together, we obtain:

Proposition 2.4.1.7. *The deformation condition $\mathcal{D}_v^{\mathrm{ram}}$ is liftable. The dimension of its tangent space L_v^{ram} equals $\dim H^0(\Gamma_v, \mathrm{ad}(\overline{\rho}))$. Any $f \in L_v^{\mathrm{ram}} \subset H^1(\Gamma_v, \mathrm{ad}(\overline{\rho}))$ satisfies $f(\mathrm{Frob}_v) \in \mathfrak{t}_\alpha$.*

Proof. Combine Lemma 2.4.1.4, 2.4.1.5, and 2.4.1.6. (The last assertion needs the additional fact, noted in the proof of Lemma 2.4.1.5, that 1-coboundaries have vanishing $\mathfrak{t}_\alpha$ component.) □

Denote the derived group of G by G', and let $S = G/G'$ with quotient map $\mu : G \to S$.. Suppose we are given a lift ν of $\mu \circ \overline{\rho} : \Gamma_v \to S(k)$. Let $\mathcal{D}_v^{\mathrm{ram},\nu}$ denote the sub-functor of $\mathcal{D}_v^{\mathrm{ram}}$ consisting of lifts $\rho : \Gamma_{K_v} \to G(A)$ such that $\nu_A = \mu \circ \rho$. This is a deformation condition (representable), as the condition $\mu \circ \rho = \nu$ cuts out a closed subscheme of the universal lifting ring for $\mathcal{D}_v^{\mathrm{ram}}$.

Corollary 2.4.1.8. *Suppose that ν is unramified. Then $\mathcal{D}_v^{\mathrm{ram},\nu}$ is liftable, and its tangent space has dimension $\dim H^0(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))$.*

Proof. By construction, all the deformations in $\mathcal{D}_v^{\mathrm{ram}}$ are tame. For a cocycle

$$f \in Z^1(\Gamma_v^{\mathrm{t}}, \mathrm{ad}(\overline{\rho})) \subset Z^1(\Gamma_v, \mathrm{ad}(\overline{\rho}))$$

remember that the associated tame lift is given by $\rho(g) = \exp_G(f(g)\epsilon)\overline{\rho}(g)$ for $g \in \Gamma_v^{\mathrm{t}}$. By functoriality of exp, we see that

$$\mu \circ \rho(g) = \exp_S((d\mu \circ f)(g)\epsilon)\, (\mu \circ \overline{\rho})\, (g).$$

But $\mu \circ \overline{\rho} = \nu$, so $\nu = \mu \circ \rho$ provided

$$\exp(d\mu(f(g))\epsilon) = 1$$

for all $g \in \Gamma_v^{\mathrm{t}}$. In other words, f must factor through $\ker d\mu = \mathrm{Lie}\, G' = \mathrm{ad}^0(\overline{\rho})$. As the characteristic is very good for G, the decomposition $\mathrm{ad}(\overline{\rho}) = \mathrm{ad}^0(\overline{\rho}) \oplus \mathfrak{z}_{\mathfrak{g}}$ gives a projection $\pi_{\mathfrak{z}_{\mathfrak{g}}} : \mathfrak{g} \to \mathfrak{z}_{\mathfrak{g}}$. The lift ρ satisfies $\nu = \mu \circ \rho$ if and only if $\pi_{\mathfrak{z}_{\mathfrak{g}}} \circ f = 0$. A basis for $H^1(\Gamma_v, \mathfrak{t}) \cap L_v^{\mathrm{ram}}$ is given by $f_1, \ldots f_{\dim t-1}$, all of which are unramified 1-cocycles. All the coboundaries are unramified, so by considering the composition of projections

$$H^1(\Gamma_v^{\mathrm{t}}, \mathrm{ad}(\overline{\rho})) \to H^1(\Gamma_v^{\mathrm{t}}, \mathfrak{t}) = H^1(\Gamma_v^{\mathrm{nr}}, \mathfrak{t}) \to H^1(\Gamma_v^{\mathrm{t}}, \mathfrak{z}_{\mathfrak{g}})$$

we see it suffices to check that $\pi_{\mathfrak{z}_{\mathfrak{g}}} \circ f|_{\Gamma_v^{\mathrm{nr}}} = 0$ (using the splitting (2.4.1.1)).

Lemma 2.4.1.5 gives an explicit description of L_v^{ram}. The cocycle f_0 certainly satisfies $\pi_{\mathfrak{z}_{\mathfrak{g}}} \circ f = 0$ as its image lies in $\mathfrak{g}_\alpha$. The cocycles $f_1, \ldots, f_{\dim t-1}$ give a basis for $\mathrm{Hom}(\Gamma_v^{\mathrm{nr}}, \mathfrak{t}_\alpha)$. As $\mathfrak{z}_{\mathfrak{g}}$ is a direct factor of $\mathfrak{t}_\alpha$, we see a codimension-$\dim \mathfrak{z}_{\mathfrak{g}}$ subspace of L_v^{ram} satisfies the deformation condition by projecting to 0 in $\mathfrak{z}_{\mathfrak{g}}$. We conclude the tangent space has dimension $h^0(\Gamma_v, \mathrm{ad}(\overline{\rho})) - \dim \mathfrak{z}_{\mathfrak{g}} = h^0(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))$.

Now consider a small surjection $A_1 \to A_0$ with $A_0 = A_1/I$ (so $\mathfrak{m}_{A_1} I = 0$), and a lift $\rho_0 \in \mathcal{D}_v^{\mathrm{ram},\nu,\square}(A_0)$. To check liftability, we may factor into a sequence of small surjections where I is a 1-dimensional k-vector space, so it suffices to treat that case. By Lemma 2.4.1.6, there exists a lift $\rho_1 \in \mathcal{D}_v^{\mathrm{ram},\square}(A_1)$, and which may be expressed as

$$\rho_1(g) = \exp_G(h(g) \otimes i)\rho_0(g)$$

for some $i \in I \simeq k$ and $h \in Z^1(\Gamma_v^{\mathrm{t}}, \mathrm{ad}(\overline{\rho}))$. (The cocycle h factors through the tame quotient as all lifts in $\mathcal{D}_v^{\mathrm{ram}}$ are tamely ramified.) We will modify h so that $\pi_{\mathfrak{z}_{\mathfrak{g}}} \circ h = 0$. To do so, pick a basis $t_1, \ldots t_r$ of $\mathfrak{z}_{\mathfrak{g}}$, and as in the proof of Lemma 2.4.1.5 consider unramified cocycles $c_j : \Gamma_v \to \Gamma_v^{\mathrm{nr}} \to \mathrm{ad}(\overline{\rho})$ defined by

$$c_j(\mathrm{Frob}_v) = t_j.$$

Let $a_j \in k$ be such that $\pi_{\mathfrak{z}_{\mathfrak{g}}} h(\mathrm{Frob}_v) = \sum_j a_j t_j$. Then because we already know $\mu \circ \rho_0 = \nu$ on A_0-points, we see

$$\exp\left(-\sum_j a_j c_j(g) \otimes i\right) \exp(h(g) \otimes i)\rho_0(g) = \exp\left(\left(h(g) - \sum_j a_j c_j(g)\right) \otimes i\right) \rho_0(g) \in \mathcal{D}_v^{\mathrm{ram},\nu,\square}(A_1)$$

because we have remarked it suffices to check $\pi_{\mathfrak{z}_{\mathfrak{g}}} \left(h(g) - \sum_j a_j c_j(g)\right) = 0$ for $g \in \Gamma_v^{\mathrm{nr}}$. Thus we may modify ρ_1 so that it is an element of $\mathcal{D}_v^{\mathrm{ram},\nu,\square}(A_1)$, and hence $\mathcal{D}_v^{\mathrm{ram},\nu}$ is liftable. □

2.4.2 Shrinking the Dual Selmer Group

Finally, we will show we can find places of K where (A1) holds, as well as some additional cohomological conditions that will allow us to shrink the dual Selmer group. Let $\overline{\rho} : \Gamma_K \to G(k)$ be a big representation, and S a finite set of places containing those above p, the archimedean places, and all places where $\overline{\rho}$ is ramified. Let $\mathcal{D}_S$ be a global deformation condition satisfying the tangent space inequality (2.2.2.2) and for which $H^1_{\mathcal{D}_S^\perp}(\Gamma_S, \mathrm{ad}^0(\overline{\rho})^*)$ is non-zero.

Lemma 2.4.2.1. *Under this assumption, the group $H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$ is non-zero..*

Proof. Let L_v be the local deformation conditions for $v \in S$. The tangent space inequality (2.2.2.2) states that

$$\sum_{v \in S} \dim L_v \geq \sum_{v \in S} \dim H^0(\Gamma_v, \mathrm{ad}^0(\overline{\rho})).$$

As $\overline{\rho}$ is big, $H^0(K, \mathrm{ad}^0(\overline{\rho}))$ and $H^0(K, \mathrm{ad}^0(\overline{\rho})^*)$ are trivial. Then the product formula (2.2.1.1) gives that

$$\dim H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) - \dim H^1_{\mathcal{D}_S^\perp}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*) = \sum_{v \in S} \left(\dim L_v - \dim H^0(\Gamma_v, \mathrm{ad}^0(\overline{\rho})\right) \geq 0$$

As we have assumed $H^1_{\mathcal{D}_S^\perp}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*)$ is non-zero, it follows that $H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$ is non-zero as well. □

Pick any non-zero element $\psi \in H^1_{\mathcal{D}_S^\perp}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*)$, and pick a non-zero $\xi \in H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$ using the Lemma. Now let F be the smallest field over K for which Γ_F is killed by $\overline{\chi}$ and $\mathrm{ad}^0(\overline{\rho})$; note that F/K is finite Galois. The absolute Galois group Γ_F acts trivially on $\mathrm{ad}^0(\overline{\rho})$ and $\mathrm{ad}^0(\overline{\rho})^*$, so the respective restrictions of ξ and ψ to $H^1(\Gamma_F, \mathrm{ad}^0(\overline{\rho}))$ and $H^1(\Gamma_F, \mathrm{ad}^0(\overline{\rho})^*)$ are just homomorphisms, which we will respectively denote by ξ' and ψ'. Let $F_{\xi'}$ and $F_{\psi'}$ be the fixed fields of the kernel of ξ' and ψ' respectively.

Lemma 2.4.2.2. *The elements ξ' and ψ' lie in $H^1(\Gamma_F, \mathrm{ad}^0(\overline{\rho}))^{\mathrm{Gal}(F/K)}$ and $H^1(\Gamma_F, \mathrm{ad}^0(\overline{\rho}))^{\mathrm{Gal}(F/K)}$ respectively, and are non-zero. Furthermore $F_{\xi'}$ and $F_{\psi'}$ are Galois over K.*

Proof. By the bigness assumption, $H^1(K(\mathrm{ad}^0(\overline{\rho}))/K, \mathrm{ad}^0(\overline{\rho})) = 0$. The extension $F/K(\mathrm{ad}^0(\overline{\rho}))$ is Galois of degree prime to p by Lemma 2.3.1.8, so the inflation-restriction sequence gives

$$0 \to H^1(\mathrm{Gal}(K(\mathrm{ad}^0(\overline{\rho}))/K), \mathrm{ad}^0(\overline{\rho})) \to H^1(\mathrm{Gal}(F/K), \mathrm{ad}^0(\overline{\rho})) \to H^1(\mathrm{Gal}(F/K(\mathrm{ad}^0(\overline{\rho})), \mathrm{ad}^0(\overline{\rho})) = 0$$

Therefore $H^1(\mathrm{Gal}(F/K), \mathrm{ad}^0(\overline{\rho})) = 0$. Another application of the inflation-restriction sequence gives

$$0 \to H^1(\mathrm{Gal}(F/K), \mathrm{ad}^0(\overline{\rho})) \to H^1(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) \to H^1(\Gamma_F, \mathrm{ad}^0(\overline{\rho}))^{\mathrm{Gal}(F/K)}$$

The first term is zero, so image $\xi' \in H^1(\Gamma_F, \mathrm{ad}^0(\overline{\rho}))^{\mathrm{Gal}(F/K)}$ is non-zero. The same argument applies to the field $K(\mathrm{ad}^0(\overline{\rho})^*)$ and the module $\mathrm{ad}^0(\overline{\rho})^*$, showing that $\psi' \in H^1(\Gamma_F, \mathrm{ad}^0(\overline{\rho}))^{\mathrm{Gal}(F/K)}$ is non-zero.

Finally, oberserve that $F_{\xi'}$ and $F_{\psi'}$ are Galois over K as the natural Γ_K-actions on $H^1(\Gamma_F, \mathrm{ad}^0(\overline{\rho}))$ and $H^1(\Gamma_F, \mathrm{ad}^0(\overline{\rho})^*)$ leave ξ' and ψ' invariant. □

We will also need the following Lemma about the $k[\Gamma_K]$-modules $\mathrm{ad}^0(\overline{\rho})$ and $\mathrm{ad}^0(\overline{\rho})^*$.

Lemma 2.4.2.3. *There exist simple $\mathbf{F}_p[\Gamma_K]$-modules W and W' such that $\mathrm{ad}^0(\overline{\rho}) = W^{\oplus r}$ and $\mathrm{ad}^0(\overline{\rho})^* = (W')^{\oplus r'}$ as $\mathbf{F}_p[\Gamma_K]$-modules. Furthermore, W and W' are not isomorphic.*

Proof. Let W and W' be simple $\mathbf{F}_p[\Gamma_K]$-submodules of $\mathrm{ad}^0(\overline{\rho})$ and $\mathrm{ad}^0(\overline{\rho})^*$. The maps $k \otimes_{\mathbf{F}_p} W \to \mathrm{ad}^0(\overline{\rho})$ and $k \otimes_{\mathbf{F}_p} W' \to \mathrm{ad}^0(\overline{\rho})^*$ are surjective as $\mathrm{ad}^0(\overline{\rho})$ and $\mathrm{ad}^0(\overline{\rho})^*$ are irreducible over k. Furthermore the sources are direct sums of copies of W and W' respectively (as $\mathbf{F}_p[\Gamma_K]$-modules). Thus there exist r and r' for which $\mathrm{ad}^0(\overline{\rho}) = W^{\oplus r}$ and $\mathrm{ad}^0(\overline{\rho})^* = (W')^{\oplus r'}$. We will consider the eigenspaces of the action of γ on W and W', and deduce that W and W' cannot be isomorphic.

The decomposition (2.4.1.2) describes how γ acts on $\mathrm{ad}^0(\overline{\rho})$ and $\mathrm{ad}^0(\overline{\rho})^*$. It acts trivially on $\mathfrak{t}$, and acts as $\alpha(\overline{\rho}(\gamma)) = \overline{\chi}(\gamma)$ on $\mathfrak{g}_\alpha$. Furthermore, there is no other root space on which γ acts by $\overline{\chi}(\gamma)$. Now consider how γ acts on W: it must have an eigenspace with eigenvalue 1 of dimension at least $\frac{1}{r}\dim_{\mathbf{F}_p}\mathfrak{t}$, and an eigenspace of eigenvalue $\overline{\chi}(\lambda)$ of dimension $\frac{1}{r}\dim_{\mathbf{F}_p}\mathfrak{g}_\alpha$. (This uses that the eigenvalues lie in $\mathbf{F}_p$: in that case the eigenspace with eigenvalue λ on $\mathrm{ad}^0(\overline{\rho})$ is a direct sum of r copies of the eigenspace with eigenvalue λ on W.) On the other hand, by decomposing $\mathrm{ad}^0(\overline{\rho})^*$ we see that γ acts as $\overline{\chi}(\gamma)$ on $\mathfrak{t}^*$ and the identity on $\mathfrak{g}_\alpha^*$, and there is no other root space on which γ acts as the identity. This implies γ acts on W' with an eigenspace of eigenvalue 1 with dimension $\frac{1}{r'}\dim_{\mathbf{F}_p}\mathfrak{g}_\alpha^*$ and with an eigenspace of eigenvalue $\overline{\chi}(\gamma)$ with dimension at least $\frac{1}{r'}\dim_{\mathbf{F}_p}\mathfrak{t}$. As we know $\overline{\chi}(\gamma) \neq 1$, as long as $\dim_k \mathfrak{t} > 1 = \dim_k \mathfrak{g}_\alpha$, this shows W and W' cannot be isomorphic.

If the dimension of the maximal torus is 1, then $\mathfrak{g}' = \mathfrak{sl}_2$. The eigenvalues of γ acting on $\mathrm{ad}^0(\overline{\rho})$ are $\overline{\chi}(\gamma), 1$, and $\overline{\chi}(\gamma)^{-1}$, while the eigenvalues of γ acting on $\mathrm{ad}^0(\overline{\rho})^*$ are $1, \overline{\chi}(\gamma)$, and $\overline{\chi}(\gamma)^2$. Since we assumed that $\overline{\chi}(\gamma)^3 \neq 1$ when the maximal torus is one-dimensional (Definition 2.3.1.1(iii)), the eigenvalues are distinct and a similar argument shows W and W' are not isomorphic. □

We can now find places v where we can define Ramakrishna's deformation problem; the key tool is the Chebotarev density theorem.

Proposition 2.4.2.4. *There exists a place $v \notin S$ such that:*

(i) assumption (A1) holds, so we obtain a liftable deformation condition $\mathcal{D}_v^{\mathrm{ram}}$ whose tangent space L_v^{ram} has dimension $\dim H^0(\Gamma_v, \mathrm{ad}(\overline{\rho}))$;

(ii) there exists $\xi \in H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$ whose restriction to $H^1(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))$ does not lie in L_v^{ram};

(iii) there exists $\psi \in H^1_{\mathcal{D}_S^\perp}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*)$ *whose restriction to* $H^1(\Gamma_v/I_v, \mathrm{ad}^0(\overline{\rho})^*)$ *is non-zero.*

Proof. We will always pick the place v so that (A1) holds. The first step is to find an additional condition on v so that $\mathrm{res}_v\,\xi$ is not in L_v^{ram}. By Proposition 2.4.1.7, it suffices to check that $\xi(\mathrm{Frob}_v) \notin \mathfrak{t}_\alpha$. We will find an element $g \in \Gamma_F$ such that this holds whenever $\mathrm{Frob}_v \in g\gamma\Gamma_{F_{\xi'}}$.

Let $\gamma \in \Gamma_K$ be the element provided by Definition 2.3.1.1(iii). We claim there exists $\overline{g} \in \mathrm{Gal}(F_{\xi'}/F)$ such that $\xi(\overline{g}\gamma) \notin \mathfrak{t}_\alpha$. If $\xi(\gamma) \notin \mathfrak{t}_\alpha$, take $\overline{g}$ to be the identity. Otherwise, consider the image of the homomorphism $\xi' : \Gamma_F \to \mathrm{ad}^0(\overline{\rho})$. By Lemma 2.4.2.2 ξ' is fixed by the action of $\mathrm{Gal}(F/K)$, so the span of the image is a representation of Γ_K. But $\mathrm{ad}^0(\overline{\rho})$ is irreducible (Definition 2.3.1.1 (i)), so the image of ξ' cannot be contained in $\mathfrak{t}_\alpha$ (as then its span would be a proper sub-representation). Thus, there exists $\overline{g} \in \mathrm{Gal}(F_{\xi'}/F)$ such that $\xi'(\overline{g}) \notin \mathfrak{t}_\alpha$. As the adjoint action of $\overline{g}$ on $\mathrm{ad}^0(\overline{\rho})$ is trivial, we compute that

$$\xi(\overline{g}\gamma) = \xi(\overline{g}) + \mathrm{ad}(\overline{\rho}(\overline{g}))\xi(\gamma) = \xi(\overline{g}) + \xi(\gamma) \notin \mathfrak{t}_\alpha.$$

Thus if $\mathrm{Frob}_v \in \overline{g}\gamma\Gamma_{F_{\xi'}}$, we see $\xi(\mathrm{Frob}_v) \notin \mathfrak{t}_\alpha$ and hence $\mathrm{res}_v\,\xi \notin L_v^{\mathrm{ram}}$.

The second step is to find a condition on v so that the image of ψ in $H^1(\Gamma_v/I_v, \mathrm{ad}^0(\overline{\rho})^*)$ is not zero. Note that I_v acts trivially on $\mathrm{ad}^0(\overline{\rho})^*$ since v does lie above p and ρ is unramified outside of S. By definition of $\mathcal{D}_S^\perp$, the composite map

$$H^1_{\mathcal{D}_S^\perp}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*) \to H^1(\Gamma_v, \mathrm{ad}^0(\overline{\rho})^*) \to H^1(I_v, \mathrm{ad}^0(\overline{\rho})^*)$$

vanishes, so by inflation-restriction we obtain a natural map

$$H^1_{\mathcal{D}_S^\perp}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*) \to H^1(\Gamma_v/I_v, \mathrm{ad}^0(\overline{\rho})^*).$$

Using Fact 2.2.1.4, the dimension of $H^1(\Gamma_v/I_v, \mathrm{ad}^0(\overline{\rho})^*)$ is the dimension of $H^0(\Gamma_v, \mathrm{ad}^0(\overline{\rho})^*)$. But as we are assuming (A1) holds, (2.4.1.2) shows the space where Frob_v acts trivially is the one-dimensional $\mathfrak{g}_\alpha^*$. We can write down a crossed homomorphism that generates it: the function sending Frob_v to a non-zero $x_\alpha^* \in \mathfrak{g}_\alpha^* \subset \mathrm{ad}^0(\overline{\rho})^*$. It is not a 1-coboundary: as Frob_v acts trivially on $\mathfrak{g}_\alpha^*$, all 1-coboundaries are zero. Any non-zero element $f \in H^1(\Gamma_v/I_v, \mathrm{ad}^0(\overline{\rho})^*)$ must be a multiple of this function, so $f(\mathrm{Frob}_v)(x_\alpha)$ will be non-zero for a fixed non-zero $x_\alpha \in \mathfrak{g}_\alpha \subset \mathrm{ad}^0(\overline{\rho})$. So we just need to arrange that $(\mathrm{res}_v\,\psi)(\mathrm{Frob}_v)(x_\alpha) \neq 0$.

Now we can use the same technique as in the first step. Lemma 2.4.2.2 shows that ψ' is a non-zero homomorphism fixed by Γ_K. We will find $\overline{g}' \in \mathrm{Gal}(F_{\psi'}/F)$ such that $\psi(\overline{g}'\gamma)(x_\alpha) \neq 0$. If $\psi(\gamma)(x_\alpha) \neq 0$, take $\overline{g}'$ to be the identity in $\mathrm{Gal}(F_{\psi'}/F)$. Suppose instead that $\psi(\gamma)(x_\alpha) = 0$. The span of the image of ψ' is a non-zero representation of Γ_K contained in the irreducible $\mathrm{ad}^0(\overline{\rho})^*$, so there must exist $\overline{g}' \in \mathrm{Gal}(F_{\psi'}/F)$ such that $\psi'(\overline{g}')(x_\alpha) \neq 0$. Then as $\overline{g}'$ acts trivially under the adjoint action (by definition of F), we compute

$$\psi(\overline{g}'\gamma)(x_\alpha) = \psi(\overline{g}')(x_\alpha) + \left(\mathrm{ad}^0(\overline{\rho})^*(\overline{g}')\right)\psi(\gamma)(x_\alpha) = \psi(\overline{g}')(x_\alpha) \neq 0$$

In either case, there exists a $\overline{g}' \in \mathrm{Gal}(F_{\psi'}/F)$ such that $\psi(\overline{g}'\gamma)(x_\alpha) \neq 0$.

The final step is to pick the prime v so that all of these conditions hold (to be found via the Chebotarev density theorem). Let g and g' denote lifts of $\overline{g}$ and $\overline{g}'$ to Γ_F. Consider the subset Σ of Γ_S consisting of elements γ' such that

1. assumption (A1) holds with γ' in the role of Frob_v there;
2. $\gamma' \in g\gamma\Gamma_{F_{\xi'}}$;
3. $\gamma' \in g'\gamma\Gamma_{F_{\psi'}}$.

We must show Σ is non-empty. We claim the first condition follows from the second. As Γ_F acts trivially on $\mathrm{ad}^0(\overline{\rho})$, we know from Lemma 2.3.1.6 that $\overline{\rho}(\Gamma_F) \subset Z_G(k) \cap \overline{\rho}(\Gamma_K)$. In particular, $\overline{\rho}(\gamma')$ is regular semisimple and $Z_{G_k}(\overline{\rho}(\gamma'))^\circ = Z_{G_k}(\overline{\rho}(\gamma))^\circ$. We chose γ as in Definition 2.3.1.1, so this is the torus T_k. By assumption, there is a unique root $\alpha \in \Phi(G_k, T_k)$ for which

$$\alpha(\overline{\rho}(\gamma')) = \alpha(\overline{\rho}(\gamma)) = \overline{\chi}(\gamma) = \overline{\chi}(\gamma')$$

using that $Z_G(k) \subset (\ker(\alpha))(k)$ and that $\Gamma_F \subset \ker(\overline{\chi})$. So it suffices to consider the second and third conditions.

Note that Σ is the preimage of its image under $\Gamma_S \twoheadrightarrow \mathrm{Gal}(L/K)$ for any finite Galois L/K containing $F_{\xi'}$ and $F_{\psi'}$. Taking L to be the compositum of $F_{\xi'}$ and $F_{\psi'}$, we obtain an injection

$$\mathrm{Gal}(L/K) \hookrightarrow \mathrm{Gal}(F_{\xi'}/K) \times_{\mathrm{Gal}(F/K)} \mathrm{Gal}(F_{\psi'}/K) \tag{2.4.2.1}$$

We claim that this is an equality, which is to say $F_{\xi'} \cap F_{\psi'} = F$. We have injections ξ' and ψ' from $\mathrm{Gal}(F_{\xi'}/F)$ and $\mathrm{Gal}(F_{\psi'}/F)$ into $\mathrm{ad}^0(\overline{\rho})$ and $\mathrm{ad}^0(\overline{\rho})^*$ respectively. For $h \in \Gamma_K$ and $h' \in \mathrm{Gal}(F_{\psi'}/F)$ we have $h\xi'(hh'h^{-1}) = \xi'(h')$ as ξ' is Γ_K-invariant. Furthermore, for $a, b \in \mathbf{Z}$ and $h_1, h_2 \in \mathrm{Gal}(F_{\xi'}/F)$ we see that

$$a\xi'(h_1) + b\xi'(h_2) = \xi'(h_1^a h_2^b).$$

Thus the image of ξ' is a non-zero $\mathbf{F}_p[\Gamma_K]$-submodule of $\mathrm{ad}^0(\overline{\rho})$, and so is a direct sum of copies of W from Lemma 2.4.2.3. As ξ' is injective, $\mathrm{Gal}(F_{\xi'}/F)$ is isomorphic to a direct sum of copies of W as $\mathbf{F}_p[\Gamma_K]$-modules. Likewise, $\mathrm{Gal}(F_{\xi'}/(F_{\xi'} \cap F_{\psi'}))$ is a possibly vanishing $\mathbf{F}_p[\Gamma_K]$-submodule. We conclude that the Γ_K-invariant quotient

$$\mathrm{Gal}(F_{\xi'}/F)/\,\mathrm{Gal}(F_{\xi'}/(F_{\xi'} \cap F_{\psi'})) \simeq \mathrm{Gal}(F_{\xi'} \cap F_{\psi'}/F)$$

is isomorphic to a direct sum of copies of W as a $\mathbf{F}_p[\Gamma_K]$-module. Similarly, we see that $\psi'(\mathrm{Gal}(F_{\psi'}/F))$ is an $\mathbf{F}_p[\Gamma_K]$-module that is isomorphic to a direct sum of copies of W', so the quotient $\mathrm{Gal}(F_{\xi'} \cap F_{\psi'}/F)$ is also isomorphic to a direct sum of copies of W'. But as W and W' are not isomorphic (Lemma 2.4.2.3), $\mathrm{Gal}(F_{\xi'} \cap F_{\psi'}/F)$ is trivial; i.e. $F = F_{\xi'} \cap F_{\psi'}$ as desired. Thus (2.4.2.1) is an isomorphism and Σ is non-empty.

Now the Chebotarev density theorem implies there exists a place v outside of S (actually infinitely many) for which $\mathrm{Frob}_v \in \Sigma$. The three conditions imply that

1. assumption (A1) holds, so we can define the condition $\mathcal{D}_v^{\mathrm{ram}}$;
2. $\mathrm{res}_v\, \xi \notin L_v^{\mathrm{ram}}$;
3. The image of ψ is non-zero in $H^1(\Gamma_v/I_v, \mathrm{ad}^0(\overline{\rho})^*)$.

This completes the proof. ∎

With $\overline{\rho}$ a big representation as before and $\mathcal{D}_S = \{\mathcal{D}_w\}_{w \in S}$ a deformation condition for $\overline{\rho}$ satisfying the tangent space inequality (2.2.2.2), allowing $\overline{\rho}$ to ramify at a place v as in Proposition 2.4.2.4 according to $\mathcal{D}_v^{\mathrm{ram}}$ will decrease the size of the dual Selmer group:

Proposition 2.4.2.5. *For $T = S \cup \{v\}$, let $\mathcal{D}_T$ be the deformation condition whose local components are $\mathcal{D}_w$ for $w \in S$ and $\mathcal{D}_v^{\mathrm{ram}}$ at v. Then*

$$\dim H^1_{\mathcal{D}_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) < \dim H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})).$$

Furthermore, $\mathcal{D}_T$ satisfies the tangent space inequality (2.2.2.2).

In the proof, the best perspective is that a deformation condition $\mathcal{D}_S$ is a collection of local deformation conditions for all places of K such that it is the unramified deformation condition at those places not in S. Then $H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$ consists of elements of $H^1(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$ whose restrictions at each place w lie in the tangent space L_w to the local deformation condition (and likewise for T in place of S).

Proof. We will first analyze what happens if we weaken $\mathcal{D}_S$ to a condition $\mathcal{D}'_T$ where there is no restriction on deformations at v. We wish to show that $H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) = H^1_{\mathcal{D}'_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$. The first is certainly a subset of the second.

Recall that as $\overline{\rho}$ is big, $H^0(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) = 0$ and $H^0(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*) = 0$. Then applying (2.2.1.1) to $\mathcal{D}'_T$ and using that the calculation of h_v^1 and h_v^0 in Lemma 2.4.1.4, we see that

$$\begin{aligned}\dim H^1_{\mathcal{D}'_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) - \dim H^1_{\mathcal{D}'^{\perp}_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*) &= \sum_{w \in T} \left(\dim L_w - \dim H^0(\Gamma_w, \mathrm{ad}^0(\overline{\rho}))\right) \\ &= 1 + \sum_{w \in S} \left(\dim L_w - \dim H^0(\Gamma_w, \mathrm{ad}^0(\overline{\rho}))\right).\end{aligned}$$

Likewise, we see that

$$\dim H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) - \dim H^1_{\mathcal{D}_S{}^\perp}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*) = \sum_{w \in S} \left(\dim L_w - \dim H^0(\Gamma_w, \mathrm{ad}^0(\overline{\rho}))\right).$$

To check $H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) = H^1_{\mathcal{D}'_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$, it suffices to show that the difference

$$\dim H^1_{\mathcal{D}'_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) - \dim H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) = 1 + \dim H^1_{\mathcal{D}'^\perp_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*) - \dim H^1_{\mathcal{D}^\perp_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*)$$

is non-positive. The distinction between $H^1_{\mathcal{D}^\perp_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*)$ and $H^1_{\mathcal{D}'^\perp_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*)$ is that the first uses the condition that the restriction at v lies in $H^1_{\mathrm{nr}}(\Gamma_v, \mathrm{ad}^0(\overline{\rho})^*)$ while the second uses the condition that it lies in $0 = H^1(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))^\perp$. The inclusion

$$\dim H^1_{\mathcal{D}'^\perp_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*) \subset \dim H^1_{\mathcal{D}^\perp_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*)$$

is strict because of the element ψ in Proposition 2.4.2.4. Thus $H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) = H^1_{\mathcal{D}'_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$.

Now we study the deformation $\mathcal{D}_T$ we actually care about. In the left exact sequence

$$0 \to H^1_{\mathcal{D}_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) \to H^1_{\mathcal{D}'_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) \to H^1(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))/L_v^{\mathrm{ram}}$$

the middle term coincides with $H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}(\overline{\rho}))$. The existence of the element ξ in Proposition 2.4.2.4 implies that

$$\dim H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) > \dim H^1_{\mathcal{D}_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})).$$

Finally, the tangent space inequality (2.2.2.2) holds for $\mathcal{D}_T$ as $\dim L_v^{\mathrm{ram}} = h_v^0$. □

Corollary 2.4.2.6. *There is a finite set of places $T \supset S$ such that the deformation condition $\mathcal{D}_T$ obtained by extending $\mathcal{D}_S$ allowing deformations according to $\mathcal{D}_v^{\mathrm{ram}}$ for $v \in T \backslash S$ satisfies*

$$H^1_{\mathcal{D}^\perp_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*) = 0.$$

Proof. We may assume that $H^1_{\mathcal{D}^\perp_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*)$ is non-zero, so Lemma 2.4.2.1 implies $H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$ is non-zero. By Proposition 2.4.2.4 and Proposition 2.4.2.5, we can choose $S_1 = S \cup \{v\}$ such that

$$\dim H^1_{\mathcal{D}_{S_1}}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) < \dim H^1_{\mathcal{D}_S}(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$$

and $\mathcal{D}_{S_1}$ satisfies the tangent space inequality (2.2.2.2). Continuing in this way, we eventually reach $T \supset S$ such that either $H^1_{\mathcal{D}^\perp_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*) = 0$ (in which case we are done) or $H^1_{\mathcal{D}_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) = 0$. By Lemma 2.4.2.1, the latter implies $H^1_{\mathcal{D}^\perp_T}(\Gamma_K, \mathrm{ad}^0(\overline{\rho})^*) = 0$ as well. □

2.5 Generalizing Ramakrishna's Method

In this section, we generalize Ramakrishna's lifting method to split connected reductive groups G defined over the ring of integers $\mathcal{O}$ in a p-adic field with residue field k. We first recall the notion of odd Galois representations, which are the class of representations to which the method applies. Then we discuss a local to global principle for lifting Galois representations subject to a global deformation condition, and finally we choose the local deformation conditions to make this possible.

2.5.1 Odd Galois Representations

We first recall the notion of a split Cartan involution. Let H be a connected reductive group over an algebraically closed field k, with Lie algebra $\mathfrak{h}$. Denote the Lie algebra of the unipotent radical of a fixed Borel subgroup of H by $\mathfrak{u}$. The following result goes back to Cartan, and in this form is [Yun14, Proposition 2.2].

Fact 2.5.1.1. *For any involution $\tau \in \mathrm{Aut}(H)$, we have $\dim \mathfrak{h}^\tau \geq \dim \mathfrak{u}$. All involutions where equality holds are conjugate under $H^{\mathrm{ad}}(k)$.*

Such involutions τ of H (or of $\mathfrak{h}$) are called *split Cartan involutions* for H.

Example 2.5.1.2. Let $H = \mathrm{GL}_2$ with $H^{\mathrm{ad}} = \mathrm{PGL}_2$ with $p \neq 2$. A split Cartan involution for H^{ad} is given by conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$: the fixed points of its action on $\mathrm{Lie}\, H^{\mathrm{ad}} = \mathfrak{sl}_2$ is the standard Cartan subalgebra, and hence is one-dimensional. On the other hand, $\dim \mathfrak{u} = 1$, so it is indeed a split Cartan involution for H^{ad}. However, conjugation by an element of $H(k)$ is never a split Cartan involution for H. Indeed, conjugation must act trivially on the 1-dimensional $\mathrm{Lie}\, Z_H$, but conjugation would also induce an automorphism of the derived group $H' = \mathrm{SL}_2$. But the fixed points of that automorphism on $\mathrm{Lie}\, H'$ are also at least 1-dimensional since $\mathfrak{sl}_2 \simeq \mathfrak{psl}_2$ as $p \neq 2$.

There are split Cartan involutions for $H = \mathrm{GL}_2$, but they are not inner automorphisms. For example, the automorphism given by transpose-inverse is a split Cartan involution for GL_2: it fixes

$$\mathrm{span} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \subset \mathfrak{gl}_2$$

Letting W denote the Weyl group, we have the following useful result about the existence of split Cartan involutions [Pat15, Lemma 4.19].

Fact 2.5.1.3. *If -1 belongs to the Weyl group W of H and the co-character $\delta^\vee$ of H^{ad} given by half the sum of the positive coroots lifts to a cocharacter of H, then $H(k)$ contains an element c such that $\mathrm{Ad}(c)$ is a split Cartan involution of $\mathrm{Lie}\, H'$. Moreover, such c can be chosen to have order 2.*

If $-1 \notin W$, then H has no inner automorphism that is a Cartan involution.

Example 2.5.1.4. This shows for example that no element of $\mathrm{PGL}_n(k)$ (or $\mathrm{GL}_n(k)$) can induce an inner automorphism that is a split Cartan involution when $n > 2$, as in those cases $-1 \notin W \simeq S_n$ inside $\mathrm{GL}_n(\mathbf{Z})$.

On the other hand, for groups of type B_n or C_n with $n \geq 2$ we have $-1 \in W$. If $n \geq 4$ is even, we also have $-1 \in W$ for type D_n. For symplectic and orthogonal *similitude* groups, the center is large enough there are no problems lifting half the sum of the positive coroots. Thus we can produce a split Cartan involution by conjugation.

For a real place v of K, let $c_v \in \Gamma_v$ be a complex conjugation (well-defined up to conjugacy). We follow [Gro] and say that a Galois representation $\overline{\rho} : \Gamma_K \to G(k)$ is *odd* at v when $\mathrm{ad}(\overline{\rho}(c_v))$ is a split Cartan involution for G^{ad} (with $\mathrm{Lie}\, G^{\mathrm{ad}} = \mathrm{ad}^0(\overline{\rho})$ since p is very good for G).

Example 2.5.1.5. Let $G = \mathrm{GL}_2$ and assume $\mathrm{char}(k) \neq 2$. Conjugation by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ induces a split Cartan involution of $\mathfrak{sl}_2$. Any element of order 2 in $\mathrm{GL}_2(k)$ with determinant -1 is conjugate to this matrix (think of the Jordan canonical form), so we recover the usual definition that $\overline{\rho} : \Gamma_K \to \mathrm{GL}_2(k)$ is odd at v when $\det \overline{\rho}(c_v) = -1$.

Example 2.5.1.6. There cannot exist odd representations for GL_n with $n > 2$ because of Example 2.5.1.4. There can exist odd representations for GSp_{2n} when $n > 1$ and for GO_m when $m \geq 5$ and $m \not\equiv 2 \pmod 4$, due to Example 2.5.1.4.

2.5.2 Local Lifting Implies Global Lifting

Let $\mathcal{D}_S = \{\mathcal{D}_v\}$ be a global deformation condition and $A_1 \to A_0$ a small extension of coefficient $\mathcal{O}$-algebras with kernel I. Note that I is a k-vector space. Consider a lift $\rho_0 : \Gamma_S \to G(A_0)$ of $\overline{\rho}$ subject to $\mathcal{D}_S$. In favorable circumstances, we can use the following local-to-global principle to produce lifts to A_1

Let G' be the derived group of G and $\mu : G \to G/G'$ the quotient. We assume that the deformation condition includes the condition of fixing a lift $\nu : \Gamma_K \to (G/G')(\mathcal{O})$ of the character $\mu \circ \overline{\rho} : \Gamma_K \to (G/G')(k)$. This means that all of the local deformation conditions have tangent spaces lying in $H^1(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))$, and the obstruction cocycles automatically land in $H^2(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))$ (see Example 2.2.2.9 and Example 2.2.2.12), with similar statements for global deformation conditions.

Proposition 2.5.2.1. *Provided $H^1_{\mathcal{D}_S^\perp}(\Gamma_S, \mathrm{ad}^0(\overline{\rho})^*) = 0$, lifting ρ_0 to A_1 subject to $\mathcal{D}_S$ is equivalent to lifting $\rho_0|_{\Gamma_v}$ to A_1 subject to $\mathcal{D}_v$ for all $v \in S$.*

Proof. One direction is obvious. Conversely, suppose we have local lifts. The key input is the Poitou-Tate exact sequence (Fact 2.2.1.7):

$$H^1(\Gamma_S, \mathrm{ad}^0(\overline{\rho})) \to \bigoplus_{v \in S} H^1(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))/L_v \to H^1_{\mathcal{D}_S^\perp}(\Gamma_S, \mathrm{ad}^0(\overline{\rho})^*)^\vee \to H^2(\Gamma_S, \mathrm{ad}^0(\overline{\rho})) \to \bigoplus_{v \in S} H^2(\Gamma_v, \mathrm{ad}^0(\overline{\rho})).$$

The vanishing of $H^1_{\mathcal{D}_S^\perp}(G_S, \mathrm{ad}^0(\overline{\rho})^*)$ implies that the first map is a surjection and the last an injection.

As $\rho_0|_{\Gamma_v}$ is liftable for all $v \in S$, the local obstructions to lifting vanish. The global obstruction to lifting ρ_0 to A_1, $\mathrm{ob}(\rho_0) \in H^2(\Gamma_S, \mathrm{ad}^0(\overline{\rho})) \otimes I$, therefore maps to 0 in $\bigoplus_{v \in S} H^2(\Gamma_v, \mathrm{ad}^0(\overline{\rho})) \otimes I$. As this latter restriction map is injective, there is a lift ρ_1 of ρ_0 to A_1 on Γ_S. We wish to show it can be chosen subject to $\mathcal{D}_S$.

The set of all lifts of $\rho_0|_{\Gamma_v}$ is an $H^1(\Gamma_v, \mathrm{ad}^0(\overline{\rho})) \otimes I$-torsor. The existence of local lifts means that there exist $\phi_v \in H^1(\Gamma_v, \mathrm{ad}^0(\overline{\rho})) \otimes I$ such that $\phi_v \cdot \rho_0|_{\Gamma_v} \in \mathcal{D}_v(A_1)$. By the surjectivity of the first map in the sequence, there exists $\phi \in H^1(\Gamma_S, \mathrm{ad}^0(\overline{\rho})) \otimes I$ such that $\phi|_v$ agrees with ϕ_v up to an element of $L_v \otimes I$ for all $v \in S$. As the set of lifts of $\rho_0|_{\Gamma_v}$ subject to $\mathcal{D}_v$ is a $L_v \otimes I$-torsor, this implies that $(\phi \cdot \rho_1)|_{\Gamma_v} \in \mathcal{D}_v(A_1)$. In other words, $\phi \cdot \rho_1$ is a lift of ρ_0 to A_1 satisfying $\mathcal{D}_S$. ☐

2.5.3 Choosing Deformation Conditions

Let G' be the derived group of G and $\mu : G \to G/G'$ be the quotient map. For a fixed lift ν of $\mu \circ \overline{\rho} : \Gamma_K \to (G/G')(k)$, the heart of the matter is to choose deformations conditions so the results of §2.5.2 produce a geometric lift of $\overline{\rho}$ with $\mu \circ \overline{\rho} = \nu$. We need:

1. Locally liftable deformation conditions at places above p whose characteristic-zero points are lattices in crystalline (or semistable representations).
2. Locally liftable deformation conditions at finite places away from p where $\overline{\rho}$ is ramified.
3. The ability to choose Ramakrishna's deformation condition (§2.4) at additional finite places away from p where $\overline{\rho}$ is unramified in order to find S so that $H^1_{\mathcal{D}_S^\perp}(\Gamma_S, \mathrm{ad}^0(\overline{\rho})^*) = 0$.

Remark 2.5.3.1. The importance of the fixed ν is discussed in §2.5.4. The technical consequence is that deformation conditions that incorporate this fixed lift can be analyzed using the Galois cohomology of $\mathrm{ad}^0(\overline{\rho})$, rather than of the larger $\mathrm{ad}(\overline{\rho})$.

It is necessary to extend $\mathcal{O}$ and k in order to define some of these deformation conditions: the condition that $\overline{\rho}$ is big is unaffected (Remark 2.3.1.3), so we are free to do so. We will find such deformation conditions when $G = \mathrm{GSp}_m$ with even $m \geq 4$ or $G = \mathrm{GO}_m^\circ$ with $m \geq 5$. In order to have the necessary oddness assumption on $\overline{\rho}$, in the latter case $m \not\equiv 2 \pmod 4$.

Remark 2.5.3.2. As our analysis in this chapter applies to connected reductive groups, we need to use the connected group GO_m°. It is possible to modify the arguments to apply to some disconnected groups (see the treatment of L-groups in [Pat15, §9]) but we do not do so here. Given $\overline{\rho} : \Gamma_K \to \mathrm{GO}_m^\circ(k) \subset \mathrm{GO}_m(k)$, viewing it as a representation for the group $G = \mathrm{GO}_m$ any deformation to a coefficient ring A will automatically factor through $\mathrm{GO}_m^\circ(A)$, so deformation conditions for the larger group GO_m naturally give deformation conditions for GO_m°.

At the places above p, when $G = \mathrm{GO}_m^\circ$ or GSp_m after extending k we will construct a *Fontaine-Laffaille deformation condition* using Fontaine-Laffaille theory in Chapter 3. This requires the assumption that $\nu \otimes \mathcal{O}[\frac{1}{p}]$ is crystalline, p is unramified in K, $\overline{\rho}$ is torsion-crystalline with Hodge-Tate weights in an interval of length $\frac{p-2}{2}$, and that the Fontaine-Laffaille weights for each $\mathbf{Z}_p$-embedding of $\mathcal{O}_K$ into $\mathcal{O}$ are pairwise distinct. The deformation condition will be proved to be liftable, and the dimension of the tangent space will be $h^0(\Gamma_v, \mathrm{ad}^0(\overline{\rho})) + [K_v : \mathbf{Q}_p] \dim_k \mathfrak{u}$, where $\mathfrak{u}$ is the Lie algebra of the unipotent radical of a Borel subgroup of G. This generalizes the results for GL_n obtained in [CHT08, §2.4.2].

Remark 2.5.3.3. The restriction that p is unramified in K and that the Hodge-Tate weights of $\overline{\rho}$ are in an interval of length $\frac{p-2}{2}$ is required to use Fontaine-Laffaille theory. Approaches using different flavors of integral p-adic Hodge theory should be able to remove it (for example, the deformation condition based on ordinary representations worked out by Patrikis [Pat15, §4.1] does so for a special class of representations). However, most previous work on studying deformation rings using integral p-adic Hodge theory only gives results about the deformation ring with p inverted, which does not suffice for our method.

The assumption that the Hodge-Tate weights are pairwise distinct is crucial, as otherwise the expected dimensions of the local crystalline deformation rings are too small to use in Ramakrishna's method.

At the places where $\overline{\rho}$ is ramified, in Chapter 4 we will construct a *minimally ramified deformation condition* by studying deformations of nilpotent (or equivalently unipotent) provided $p \geq m$. For each place, this will potentially require a finite extension of k. After such a further extension, this will be a liftable deformation condition at v with tangent space of dimension $h^0(\Gamma_v, \mathrm{ad}^0(\overline{\rho}))$ (see Corollary 4.5.3.4). This generalizes the results for GL_n obtained in [CHT08, §2.4.4].

We also need to specify a deformation condition at the archimedean places v: we just require lifts for which $\mu \circ \rho|_{\Gamma_v} = \nu|_{\Gamma_v}$. This condition is very simple to arrange, as $\#\Gamma_v \leq 2$. At a complex place, the dimension of the tangent space is zero and the dimension of the invariants is $\dim_k \mathrm{ad}^0(\overline{\rho})$. At a real place, the tangent space is zero when $p > 2$ and the invariants are the invariants of complex conjugation on $\mathrm{ad}^0(\overline{\rho})$.

In order to construct Ramakrishna's deformation condition, the tangent space inequality (2.2.2.2) must be satisfied. Let S be a set of places consisting of primes above p, places where $\overline{\rho}$ is ramified, and the archimedean places. When using the local deformation conditions as above at $s \in S$, the inequality (2.2.2.2) says exactly that

$$[K : \mathbf{Q}] \dim_k \mathfrak{u} = \sum_{v|p} [K_v : \mathbf{Q}_p] \dim_k \mathfrak{u} \geq \sum_{v|\infty} h^0(\Gamma_v, \mathrm{ad}^0(\overline{\rho})) = \sum_{v|\infty} \mathrm{ad}^0(\overline{\rho})^{\Gamma_v} \tag{2.5.3.1}$$

This is very strong: $\dim \mathrm{ad}^0(\overline{\rho})^{\Gamma_v} \geq [K_v : \mathbf{R}] \dim_k \mathfrak{u}$ by Fact 2.5.1.1, so (2.5.3.1) holds if and only if K is totally real and $\overline{\rho}$ is odd at all real places of K.

Assuming K is totally real and $\overline{\rho}$ is odd and at all real places, we can allow ramification at additional places to define a deformation condition to which we can apply Proposition 2.5.2.1. In particular, using Ramakrishna's deformation condition $\mathcal{D}_v^{\mathrm{ram}}$ at a collection of new places as in Corollary 2.4.2.6 (again possibly extending k), we obtain a new deformation condition $\mathcal{D}_T$ for which $H^1_{\mathcal{D}_T^\perp}(\Gamma_T, \mathrm{ad}^0(\overline{\rho})^*) = 0$. Using Proposition 2.5.2.1, we obtain the desired lifts.

Let us collect together all of our assumptions and record the result. For $G = \mathrm{GSp}_m$ with even $m \geq 1$ or $G = \mathrm{GO}_m^\circ$ with $m \geq 5$ and a big representation $\overline{\rho} : \Gamma_K \to G(k)$, fix a lift $\nu : \Gamma_K \to (G/G')(k)$ to $\mathcal{O}$ of $\mu \circ \overline{\rho}$ such that $\nu \otimes \mathcal{O}[\frac{1}{p}]$ is Fontaine-Laffaille. We furthermore assume that K is totally real and that $\overline{\rho}$ is odd at all real places (which requires $m \not\equiv 2 \pmod 4$). To use the Fontaine-Laffaille condition, we assume that p is unramified in K and that $\overline{\rho}$ is Fontaine-Laffaille at all places above p with Fontaine-Laffaille weights in an interval of length $\frac{p-2}{2}$, pairwise distinct for each $\mathbf{Q}_p$ embedding of K into $\mathcal{O}[\frac{1}{p}]$. In order to use the minimally ramified deformation condition of §4.5, we require that $p \geq m$. We extend $\mathcal{O}$ (and k) so that all of the required deformation conditions may be defined.

Theorem 2.5.3.4. *Under these conditions, there is a finite set T of places containing the archimedean places, the places above p, and the places where $\overline{\rho}$ is ramified such that there exists a lift $\rho : \Gamma_K \to G(\mathcal{O})$ such that*

- $\mu \circ \rho = \nu$.
- *ρ is ramified only at places in T*
- *ρ is Fontaine-Laffaille at all places above p, and hence crystalline.*

In particular, ρ is geometric. If we combine this with Proposition 2.3.1.4, we obtain Theorem 1.1.2.2.

Remark 2.5.3.5. Using the local deformation conditions for GL_n in [CHT08, §2.4.1] and [CHT08, §2.4.4], the same argument gives an identical result with $G = \mathrm{GL}_n$. But for $n > 2$ it is impossible to satisfy the oddness hypothesis. For GL_2, this is a variant of [Ram02, Theorem 1b].

Remark 2.5.3.6. For other groups, the method will produce lifts provided appropriate local conditions exist. The deformation conditions we used are only available in full strength for symplectic and orthogonal groups. An alternative deformation condition above p is the ordinary deformation condition [Pat15, §4.1], available for any G. For ramified primes not above p, §4.4 provides a deformation condition assuming a certain nilpotent centralizer is smooth and $\overline{\rho}|_{\Gamma_v}$ is tamely ramified of the special type considered in §4.4.

2.5.4 Lifting Representations valued in Tori

We will briefly explain the importance of fixing a lift of $\overline{\nu} = \mu \circ \overline{\rho} : \Gamma_K \to (G/G')(k)$. Pragmatically, we do so in order to use $\mathrm{ad}^0(\overline{\rho})$ instead of the larger $\mathrm{ad}(\overline{\rho})$ in Proposition 2.5.2.1 and connect with Corollary 2.4.2.6. This really is necessary however, as the universal deformation ring for $\overline{\nu}$ need not be smooth. In particular, if we have no control on the homomorphism $\Gamma_K \to G(R) \to (G/G')(R)$ induced by lifts of $\overline{\rho}$, we might not be able to lift at each step in Proposition 2.5.2.1 because there would be an obstruction to lifting $\mu \circ \rho$.

In [Til96, §4], for a split $\mathcal{O}$-torus T' in place of G/G' Tilouine proves that the universal deformation ring for $\overline{\nu}$ is isomorphic to $\mathcal{O}[[X^*(T') \otimes \Gamma_S^{\mathrm{ab},p}]]$, where $\Gamma_S^{\mathrm{ab},p}$ is the maximal abelian pro-p quotient of Γ_S. The idea is as follows. Reduce to the case of $T' = \mathbf{G}_m$ and consider the Teichmuller lift. All other lifts to a coefficient ring R differ from this fixed lift by a continuous homomorphism $\Gamma_S \to 1 + \mathfrak{m}_R$, which must factor through the maximal abelian p-quotient of Γ_S. So a lift gives a homomorphism

$$\mathcal{O}[\Gamma_S^{\mathrm{ab},p}] \to R.$$

Taking into account the topology on $\mathcal{O}[[\Gamma_S^{\mathrm{ab},p}]]$, we see that it continuously extends. Conversely, a homomorphism from $\mathcal{O}[[\Gamma_S^{\mathrm{ab},p}]]$ gives a continuous lift. Thus the universal deformation ring is $\mathcal{O}[[\Gamma_S^{\mathrm{ab},p}]]$.

Next, we briefly recall what class field theory tells us about the structure of $\Gamma_S^{\mathrm{ab},p}$. As before, we assume that S contains all primes above p and the archimedean places. Define

$$U_{p,1} = \{x \in \mathcal{O}_K^\times : x \equiv 1 \pmod{v} \text{ for all } v|p\}$$

and denote the closure of its image in $\prod_{v|p} \mathcal{O}_v^\times$ by $\overline{U}_{p,1}$. The $\mathbf{Z}_p$-rank of $\overline{U}_{p,1}$ is $r_1 + r_2 - 1 - \delta$, and Leopoldt's conjecture is that $\delta = 0$. Class field theory shows (as sketched in [Til96, §4.2]) that the group $\Gamma_S^{\mathrm{ab},p}$ is the product of a finite p-group and $\mathbf{Z}_p^{r_2+1+\delta}$. In particular, the universal deformation ring is smooth (a power series ring) when the finite part of $\Gamma_S^{\mathrm{ab},p}$ is trivial. In that situation, we could use deformation conditions without fixing a lift of $\overline{\nu}$, but in general we must fix a lift to avoid this issue.

Remark 2.5.4.1. There is always a lift of $\overline{\nu}$: as G/G' is split, we can reduce to the case of $\mathbf{G}_m$ and compose with the Teichmuller character $k^\times \to \mathcal{O}^\times$ to produce a lift. However, such a lift will not be crystalline above p as it has finite (typically ramified) image: crystalline characters are algebraic on inertia [Con14, Proposition B.4], which means such characters are either unramified or have infinite image. It is not immediately obvious that $\overline{\nu}$ will lift to characteristic zero in such a way that it is Fontaine-Laffaille at places above p, but such a condition is an obviously necessary condition for Theorem 2.5.3.4 to hold so we do not further explore this question here.

It is worth explaining why there is no *local* obstruction to $\overline{\nu}|_{\Gamma_v}$ having a Fontaine-Laffaille lift for v a place above p. Let κ be the residue field of K_v, and let L be a p-adic field with residue field k which splits K over $\mathbf{Q}_p$. Consider a crystalline character $\Gamma_{K_v} \to \mathcal{O}_L^\times$: composing with the local reciprocity homomorphism and using [Con14, Proposition B.4], on inertia this gives a map $f : \mathcal{O}_{K_v}^\times \to \mathcal{O}_L^\times$ which is induced by a $\mathbf{Q}_p$-homomorphism

$$R_{K_v/\mathbf{Q}_p}(\mathbf{G}_m) \to R_{L/\mathbf{Q}_p}(\mathbf{G}_m)$$

between Weil restrictions.

By adjunction, such homomorphisms are equivalent to L-homomorphisms

$$R_{K_v/\mathbf{Q}_p}(\mathbf{G}_m) \otimes_{\mathbf{Q}_p} L \to \mathbf{G}_m.$$

As L splits K_v over $\mathbf{Q}_p$, this is a homomorphism

$$\prod_{\tau: K_v \hookrightarrow L} (\mathbf{G}_m)_\tau \to \mathbf{G}_m.$$

Thus the map $f : \mathcal{O}_{K_v}^\times \to \mathcal{O}_L^\times$ is of the form

$$x \mapsto \prod_{\tau : K_v \hookrightarrow L} \tau(x)^{n_\tau}. \tag{2.5.4.1}$$

Fix a $\mathbf{Q}_p$-embedding τ of K_v into L and hence an embedding of κ into k: any other embedding of κ differs by a power of the Frobenius (and recall that $K_v/\mathbf{Q}_p$ is unramified since we assume p is unramified in K). By appropriate choice of the exponents $n_\tau \in \{0, 1, \ldots, p-1\}$, it is clear that any map of the form

$$\kappa^\times \ni x \mapsto x^{a_0 + pa_1 + \ldots + p^{r-1} a_{r-1}} \in k^\times$$

with $r = [k : \kappa]$ and $0 \leq a_i < p$ is the reduction of something of the form (2.5.4.1).

As $k^\times$ and $\kappa^\times$ are cyclic, any character $\chi_v : \mathcal{O}_{K_v}^\times \to k^\times$ is the reduction of a character $\nu_v : \mathcal{O}_{K_v}^\times \to \mathcal{O}_L^\times$ of the form (2.5.4.1). Thus any $\overline{\nu}_v : \Gamma_{K_v} \to k^\times$ agrees on inertia with the inertial restriction of the reduction of a crystalline character. The quotient is an unramified character, which may be lifted using a Teichmuller lift. Combining these lifts, we obtain a character $\nu_v : \Gamma_{K_v} \to \mathcal{O}_L^\times$ lifting $\overline{\nu}|_{\Gamma_v}$ which is algebraic on inertia, hence crystalline. The Hodge-Tate weights are the n_τ, which may be taken to be in an interval of length $p-1$. In particular, ν_v is Fontaine-Laffaille. (The statement of Fontaine-Laffaille theory in Chapter 3 require an interval of length $p-2$ to obtain a clean statement: in [FL82] an interval of length $p-1$ is fine as long as an additional technical condition holds: this is satisfied in our case as long as not all of the a_i are $p-1$.)

Chapter 3

Fontaine-Laffaille Deformations with Pairings

Let K be a finite extension of $\mathbf{Q}_p$, k a finite field of characteristic p, and G be a split connected reductive group over the valuation ring $\mathcal{O}$ of a p-adic field L with residue field k such that L splits K over $\mathbf{Q}_p$. Consider a residual representation $\overline{\rho} : \Gamma_K \to G(k)$. The methods of Chapter 2 require a nice class of deformations of $\overline{\rho}$. In particular, we need a deformation condition $\mathcal{D}_{\overline{\rho}}$ on the local Galois group Γ_K such that:

- $\mathcal{D}_{\overline{\rho}}$ is liftable;
- $\mathcal{D}_{\overline{\rho}}$ is large enough, in the sense that its tangent space has dimension

$$[K : \mathbf{Q}_p] \dim_k \mathfrak{u} + \dim_k H^0(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$$

  where $\mathfrak{u}$ is the Lie algebra of the unipotent radical of a Borel of G;
- $\mathcal{D}_{\overline{\rho}}(\mathcal{O})$ consists of crystalline representations (more precisely, lattices in crystalline representations).

Fontaine-Laffaille theory provides a mechanism to study deformations when p is unramified in K (so L splits K if and only if k contains the residue field of K) and the representation $\overline{\rho}$ is torsion-crystalline with Hodge-Tate weights in an interval of length $p-2$. For $G = \mathrm{GL}_n$, such a deformation condition was constructed in [CHT08, §2.4.1] provided that the Fontaine-Laffaille weights for each $\mathbf{Q}_p$-embedding $K \hookrightarrow L$ are pairwise distinct (see Remark 3.1.2.6). We need to adapt those ideas to symplectic and orthogonal groups under the additional assumption that the Fontaine-Laffaille weights lie in an interval of length $\frac{p-2}{2}$. For symplectic groups and $K = \mathbf{Q}_p$, this was addressed in Patrikis' senior thesis [Pat06]: this chapter is a mild generalization. Keeping track of the pairing requires knowledge about how Fontaine-Laffaille modules interact with duality and tensor products, the details of which are recorded in [Pat06]. As this is not readily available, we will record proofs.

In §3.1 we will review Fontaine-Laffaille theory and how it interacts with tensor products and duality. The Fontaine-Laffaille deformation condition $D_{\overline{\rho}}^{\mathrm{FL}}$ will be defined in §3.2: Theorem 3.2.1.2 shows it satisfies the three properties above needed in Chapter 2.

3.1 Fontaine-Laffaille Theory with Pairings

We begin by establishing some notation and reviewing the key results of Fontaine-Laffaille theory. It provides a mechanism to relate torsion-crystalline representations to semi-linear algebra data (Fontaine-Laffaille modules), just as p-adic Hodge theory relates crystalline representations to admissible filtered φ-modules. It was first studied by Fontaine and Laffaille [FL82], who introduced a contravariant functor relating torsion-crystalline representations and Fontaine-Laffaille modules. For deformation theory, in particular compatibility with tensor products, it is necessary to use a covariant version, introduced in [BK90]. The details of relating this covariant functor to the functor studied by Fontaine and Laffaille that are omitted in [BK90] are written down in [Con94]. We then studying Fontaine-Laffaille modules with the extra data of a pairing by analyzing tensor products and duals.

3.1.1 Covariant Fontaine-Laffaille Theory

Let $K = W(k')[\frac{1}{p}]$ for a perfect field k' of characteristic p. Let $W = W(k')$ and $\sigma : W \to W$ denote the Frobenius map. Recall that there is an equivalence of categories between $\mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Q}_p}(\Gamma_K)$ (crystalline representations of Γ_K) and $\mathrm{MF}_K^{\varphi,\mathrm{ad}}$ (admissible filtered φ-modules). For a crystalline representation V and an admissible filtered φ-module M, quasi-inverse contravariant functors are given by

$$D^*_{\mathrm{cris}}(V) = \mathrm{Hom}_{\mathbf{Q}_p[\Gamma_K]}(V, B_{\mathrm{cris}}) \quad \text{and} \quad V^*_{\mathrm{cris}}(M) = \mathrm{Hom}_{\mathrm{MF}_K^{\varphi}}(M, B_{\mathrm{cris}}).$$

Fontaine and Laffaille construct a generalization of this functor that applies to $\mathbf{Z}_p[\Gamma_K]$-modules that arise as subquotients of lattices in crystalline representations, provided the analogue of the Hodge-Tate weights of the representation lie in an interval of length $p-2$. We want a generalization of the *covariant* functors

$$D_{\mathrm{cris}}(V) = (B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)^{\Gamma_K} \quad \text{and} \quad V_{\mathrm{cris}}(M) = \mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_K M)^{\phi=1}.$$

Our convention will be that the Hodge-Tate weight of the cyclotomic character is -1, which will work well with these covariant functors.

Definition 3.1.1.1. A *torsion-crystalline* representation with Hodge-Tate weights in $[a, b]$ is a $\mathbf{Z}_p[\Gamma_K]$-module T for which there exists a crystalline representation V with Hodge-Tate weights in $[a, b]$ and Γ_K-stable lattices $\Lambda \subset \Lambda'$ in V such that Λ'/Λ is isomorphic to T.

The analogue of torsion-crystalline representations on the semilinear algebra side are certain classes of Fontaine-Laffaille modules:

Definition 3.1.1.2. A *Fontaine-Laffaille* module is a W-module M together with a decreasing filtration $\{M^i\}_{i\in\mathbf{Z}}$ of M by W-submodules and a family of σ-semilinear maps $\{\varphi^i_M : M^i \to M\}$ such that:

- The filtration is separated and exhaustive: $M = \cup_{i\in\mathbf{Z}} M^i$ and $\cap_{i\in\mathbf{Z}} M^i = 0$.
- For $m \in M^{i+1}$, $p \cdot \varphi^{i+1}_M(m) = \varphi^i_M(m)$.

Morphisms of Fontaine-Laffaille modules $f : M \to N$ are W-linear maps such that $f(M^i) \subset N^i$ and $f \circ \varphi^i_M = \varphi^i_N \circ f$ for all i. The category is denoted MF_W.

Remark 3.1.1.3. Jumps in the filtration will turn out to correspond Hodge-Tate weights, so the condition $M^a = M$ and $M^{b+1} = 0$ with $a \le b$ corresponds to restricting the Hodge-Tate weights to a certain range. We call the set of jumps in the filtration the *Fontaine-Laffaille weights*. We will denote the full subcategory of Fontaine-Laffaille modules with this additional condition by $\mathrm{MF}_W^{[a,b]}$.

We are interested in torsion Fontaine-Laffaille modules that satisfy a version of an admissibility condition.

Definition 3.1.1.4. Let $\mathrm{MF}^f_{W,\mathrm{tor}}$ denote the full subcategory of MF_W consisting of M for which M is of finite length (as a W-module) and for which $\sum_{i\in\mathbf{Z}} \varphi^i(M^i) = M$. Let $\mathrm{MF}^{f,[a,b]}_{W,\mathrm{tor}}$ denote the full subcategory with the additional condition that $M^a = M$ and $M^{b+1} = 0$.

Fact 3.1.1.5. *The category $\mathrm{MF}^{f,[a,b]}_{W,\mathrm{tor}}$ is abelian, with kernel, cokernel, and image formed as in the underlying category of W-modules. For an object $M \in \mathrm{MF}^{f,[a,b]}_{W,\mathrm{tor}}$, each M^i is a direct summand of M as a W-module.*

The first statement is [FL82, Proposition 1.8], the second is [Win84, Proposition 1.4.1 (ii)].

We are also interested in a variant that allows non-torsion modules.

Definition 3.1.1.6. A *filtered Dieudonné* module M is a Fontaine-Laffaille module for which the M^i are direct summands of M as W-modules and for which

$$\sum_{i\in\mathbf{Z}} \varphi^i(M^i) = M.$$

Let $\mathcal{D}_K$ denote the full subcategory of MF_W consisting of filtered Dieudonné modules M for which $M^a = M$ and $M^{b+1} = 0$ for some $0 \le b - a \le p - 2$.

Note that $\mathrm{MF}_{W,\mathrm{tor}}^{f,[a,b]}$ is a full subcategory of $\mathcal{D}_K$. There are natural notions of tensor products and duality.

Definition 3.1.1.7. For Fontaine-Laffaille modules M_1 and M_2, define $M_1 \otimes M_2$ to have underlying W-module $M_1 \otimes M_2$, filtration given by $(M_1 \otimes M_2)^n = \sum_{i+j=n} M_1^i \otimes M_2^j$, and maps $\varphi^n_{M_1 \otimes M_2}$ induced by the $\varphi^i_{M_1}$ and $\varphi^j_{M_2}$.

Definition 3.1.1.8. For $M \in \mathrm{MF}^f_{W,\mathrm{tor}}$, define M^* to be $\mathrm{Hom}_W(M, K/W)$ with the dual filtration $M_i^* = \mathrm{Hom}_W(M/M^{1-i}, K/W)$ and with $\varphi^i_{M^*}$ characterized for $f \in M_i^*$ and $m \in M^j$ by $\varphi^i_{M^*}(f)(\varphi^j(m)) = 0$ when $j \geq 1-i$ and by $\varphi^i_{M^*}(f)(\varphi^j(m)) = f(p^{-i-j}m)$ when $j < 1-i$ (in which case $-i-j \geq 0$).

Lemma 3.1.1.9. *There is a unique* $(\varphi^i_{M^*})$ *satisfying these constraints. Using it,* M^* *is an object of* $\mathrm{MF}^f_{W,\mathrm{tor}}$. *Then* $M \mapsto M^*$ *is a contravariant functor from* $\mathrm{MF}^f_{W,\mathrm{tor}}$ *to itself, and* $M \simeq M^{**}$ *naturally in* M.

Proof. Uniqueness is immediate, while existence is checked in [Con94, §7.5]. We will use a similar argument in Lemma 3.1.3.2 and Lemma 3.1.3.3. ☐

Remark 3.1.1.10. The indexing of the dual filtration is that used in [Fon82, §3.2] and elsewhere. Note that duality sends $\mathrm{MF}_{W,\mathrm{tor}}^{f,[a,b]}$ to $\mathrm{MF}_{W,\mathrm{tor}}^{f,[-b,-a]}$.

To connect Fontaine-Laffaille modules and torsion-crystalline representations, we use the period ring A_{cris}. For our purposes, all that is important is that A_{cris} is a period ring that has an action of Γ_K, a ring endomorphism φ (coming from the pth power map) and a filtration $\{\mathrm{Fil}^i A_{\mathrm{cris}}\}$. In particular, it carries both an action of Γ_K and the structure of a Fontaine-Laffaille module. A convenient reference is [Hat, §2.2], which reviews A_{cris} for the purposes of constructing the contravariant Fontaine-Laffaille functor. We use it to define an analogue of V_{cris}:

Definition 3.1.1.11. For $M \in \mathrm{MF}_{W,\mathrm{tor}}^{f,[2-p,1]}$, define

$$T_{\mathrm{cris}}(M) := \ker\left(1 - \varphi^0_{A_{\mathrm{cris}} \otimes M} : \mathrm{Fil}^0(A_{\mathrm{cris}} \otimes M) \to A_{\mathrm{cris}} \otimes M\right).$$

Remark 3.1.1.12. A small argument (see [Hat, §2.2]) also shows that

$$A_{\mathrm{cris},\infty} := A_{\mathrm{cris}} \otimes_W K/W = \varinjlim_n A_{\mathrm{cris}}/p^n A_{\mathrm{cris}} \in \mathrm{MF}_{W,\mathrm{tor}}^{f,[0,p-1]}.$$

This allows us define a contravariant functor from $\mathrm{MF}_{W,\mathrm{tor}}^{f,[0,p-1]}$ to $\mathrm{Rep}_{\mathbf{Z}_p}(\Gamma_K)$ by

$$T^*_{\mathrm{cris}}(M) := \mathrm{Hom}_{\mathrm{MF}_W}(M, A_{\mathrm{cris},\infty}).$$

This is well-defined as $\mathrm{Hom}_{\mathrm{MF}_W}(M, A_{\mathrm{cris},\infty})$ is $\mathbf{Z}_p$-finite (for example, using a contravariant version of Fact 3.1.1.14(4)). This functor agrees with the functor U_S considered by Fontaine and Laffaille [Hat, Remark 2.7].

If $M \in \mathrm{MF}_{W,\mathrm{tor}}^{f,[2-p,1]}$ is killed by p, then

$$\begin{aligned} T^*_{\mathrm{cris}}(M^*) &= \mathrm{Hom}_{\mathrm{MF}_W}(M^*, A_{\mathrm{cris}}/pA_{\mathrm{cris}}) \\ &\simeq \ker\left(1 - \varphi^0_{A_{\mathrm{cris}} \otimes M} : \mathrm{Fil}^0(A_{\mathrm{cris}} \otimes M) \to A_{\mathrm{cris}} \otimes M\right) \\ &= T_{\mathrm{cris}}(M) \end{aligned}$$

which is how Fontaine and Laffaille's results about T^*_{cris} imply results about T_{cris}.

We can extend T_{cris} to $\mathcal{D}_K$ by defining an analogue of Tate-twisting:

Definition 3.1.1.13. For $M \in \mathrm{MF}_{W,\mathrm{tor}}^{f,[a,b]}$ and an integer s, define $M(s)$ to have the same underlying W-module with filtration $M(s)^i = M^{i-s}$ and maps $\varphi^i_{M(s)} = \varphi^{i-s}_M$.

Tate-twisting allows us to shift Hodge-Tate weights and extend results in the range $[2-p, 1]$ to any interval $[a, b]$ where $b - a \leq p - 2$. For $M \in \mathrm{MF}_{W,\mathrm{tor}}^{f,[a,b]}$, we define

$$T_{\mathrm{cris}}(M) = T_{\mathrm{cris}}(M(-(b-1)))(b-1)$$

Fact 3.1.1.14. *We have:*

1. *The covariant functor* $T_{\mathrm{cris}} : \mathcal{D}_K \to \mathrm{Rep}_{\mathbf{Z}_p}[\Gamma_K]$ *is well-defined, and is exact and fully faithfully.*
2. *For* $M \in \mathcal{D}_K$*,* $T_{\mathrm{cris}}(M) = \varprojlim T_{\mathrm{cris}}(M/p^n M)$*.*
3. *The essential image of* $T_{\mathrm{cris}} : \mathrm{MF}^{f,[a,b]}_{W,\mathrm{tor}} \to \mathrm{Rep}_{\mathbf{Z}_p}[\Gamma_K]$ *is stable under formation of sub-objects and quotients.*
4. *For* $M \in \mathrm{MF}^{f,[a,b]}_{W,\mathrm{tor}}$*, the length of* M *as a* W*-module is equal to the length of* $T_{\mathrm{cris}}(M)$ *as a* $\mathbf{Z}_p$*-module.*
5. *For* $M \in \mathcal{D}_K$*, the* Γ_K*-representation* $T_{\mathrm{cris}}(M)[\frac{1}{p}]$ *is crystalline.*
6. *Any torsion-crystalline* $\mathbf{F}_p[\Gamma_K]$*-module* $\overline{V}$ *whose Hodge-Tate weights lie in an interval of length* $p-2$ *is in the essential image of* T_{cris}*.*

Remark 3.1.1.15. While a torsion representation does not have Hodge-Tate weights, according to Definition 3.1.1.1 the Hodge-Tate weights of a torsion-crystalline representation lie in an interval $[a, b]$ provided the representation is quotient of Γ_K-stable lattices in a crystalline representation with Hodge-Tate weights in $[a, b]$.

This is a modified version of [BK90, Theorem 4.3]. Additional details of the proof of that theorem are recorded in [Con94, §7]. The first, fourth, and fifth statements are stated explicitly in [BK90, Theorem 4.3]. The second is proven in [Con94, §7.2]. The claim about the essential image follows from the results of [Con94, §8.3-9.6]: a formal argument shows that if T^*_{cris} takes simple objects to simple objects, the essential image is stable under formation of sub-objects and quotients. The content is that T^*_{cris} takes simple objects to simple objects. The formal argument adapts to T_{cris}, and Remark 3.1.1.12 allows us to deduce that T_{cris} takes simple objects to simple objects as all simple objects are automatically killed by p.

For the last statement, we need a fact about T^*_{cris}.

Fact 3.1.1.16. *For* $r \in \{0, 1, \ldots, p-2\}$*, the functor* T^*_{cris} *induces an anti-equivalence between* $\mathrm{MF}^{f,[0,r]}_{W,\mathrm{tor}}$ *and the full subcategory of* $\mathrm{Rep}_{\mathbf{Z}_p}(\Gamma_K)$ *consisting of torsion-crystalline* Γ_K *representations with Hodge-Tate weights in* $[-r, 0]$*.*

Dualizing $\overline{V}$ and Tate-twisting, we may assume that the Hodge-Tate weights of $\overline{V}^*$ lie in $[0, p-2]$. Then $\overline{V}^* = T^*_{\mathrm{cris}}(M)$ for some $M \in \mathrm{MF}^{f,[0,p-2]}_{W,\mathrm{tor}}$. Remark 3.1.1.12 shows that $T_{\mathrm{cris}}(M^*) = \overline{V}$ using double duality (Lemma 3.1.1.9).

Remark 3.1.1.17. Our convention that the Hodge-Tate weight of the cyclotomic character is -1 makes the Fontaine-Laffaille weights and Hodge-Tate weights match under T_{cris}.

3.1.2 Tensor Products and Freeness

We now address two properties of T_{cris} where it is crucial to be using the covariant functor. Definition 3.1.1.7 defined a tensor product for Fontaine-Laffaille modules. If $M_1 \in \mathrm{MF}^{f,[a_1,b_1]}_{W,\mathrm{tor}}$ and $M_2 \in \mathrm{MF}^{f,[a_2,b_2]}_{W,\mathrm{tor}}$, it is straightforward to verify that $M_1 \otimes M_2 \in \mathrm{MF}^{f,[a_1+a_2,b_1+b_2]}_{W,\mathrm{tor}}$. The functor T_{cris} is compatible with tensor products in the following sense:

Fact 3.1.2.1. *Suppose that* M_1*,* M_2*, and* $M_1 \otimes M_2$ *each has Fontaine-Laffaille weights in an interval of length at most* $p-2$*. Then* $T_{\mathrm{cris}}(M_1) \otimes_{\mathbf{Z}_p} T_{\mathrm{cris}}(M_2) \simeq T_{\mathrm{cris}}(M_1 \otimes M_2)$*.*

There is a natural from the left to the right coming from the multiplication of A_{cris}. To check this map is an isomorphism, one first checks on simple M_1 and M_2 using Fontaine and Laffaille's classification of simple Fontaine-Laffaille modules when the residue field k' is algebraically closed. This is explained in [Con94, §10.6]. Then one uses a dévissage argument to reduce to the general case, as explained in [Con94, §7.11].

Remark 3.1.2.2. An analogue of this compatibility is stated in [FL82, Remarques 6.13(b)] for the contravariant functor T^*_{cris}. For T^*_{cris}, it only makes sense if M_1 and M_2 are killed by p; in that case

$$T^*_{\mathrm{cris}}(M_1) = \mathrm{Hom}_{\mathrm{MF}_W}(M_1, , A_{\mathrm{cris},\infty}) = \mathrm{Hom}_{\mathrm{MF}_W}(M_1, A_{\mathrm{cris}}/pA_{\mathrm{cris}})$$

and likewise for M_2. Then multiplication on $A_{\mathrm{cris}}/pA_{\mathrm{cris}}$ gives a natural map $T^*_{\mathrm{cris}}(M_1) \otimes T^*_{\mathrm{cris}}(M_2) \to T^*_{\mathrm{cris}}(M_1 \otimes M_2)$ which can be checked to be an isomorphism by dévissage. But $A_{\mathrm{cris},\infty}$ is not a ring, so there is no natural map without a p-torsion hypothesis on M_1 and M_2. This obstructs an analogue of dévissage to the p-torsion case for T^*_{cris}, and explains why it is crucial to work with the covariant functor T_{cris}.

For $M \in \mathrm{MF}^{f,[a,b]}_{W,\mathrm{tor}}$, if $V = T_{\mathrm{cris}}(M)$ has "extra structure" then so does M. For example, if V were a deformation of a residual representation over a finite field k, V would be an $\mathcal{O} = W(k)$-module. As T_{cris} is covariant and fully faithful, it is immediate that M is naturally an $\mathcal{O}$-module. The actions of $\mathbf{Z}_p$ on M via the embeddings into $\mathcal{O}$ and $W = W(k')$ are obviously compatible.

Representations of Γ_K defined over a finite extension L of $\mathbf{Q}_p$ can be viewed as $\mathbf{Q}_p$-vector spaces with the additional action of L. Assume there exists an embedding of K into L over $\mathbf{Q}_p$, so L splits K over $\mathbf{Q}_p$. These representations are modules over $L \otimes_{\mathbf{Q}_p} K \simeq \prod_{\tau : K \hookrightarrow L} L_\tau$ via $a \otimes b \mapsto (a\tau(b))$. For each $\mathbf{Q}_p$-embedding τ, there is a collection of Hodge-Tate weights.

We will generalize this structure to our setting. Now assume k' is finite, and more specifically that k' embeds in k, so $\mathcal{O}[\frac{1}{p}]$ splits the finite unramified K over $\mathbf{Q}_p$. Hence

$$\mathcal{O} \otimes_{\mathbf{Z}_p} W \simeq \prod_{\tau : W \hookrightarrow \mathcal{O}} \mathcal{O}_\tau$$

as $\mathcal{O}$-algebras, where τ varies over $\mathbf{Z}_p$-embeddings of W into $\mathcal{O}$ and W acts on $\mathcal{O}_\tau$ via τ. We likewise obtain a decomposition of the $\mathcal{O} \otimes_{\mathbf{Z}_p} W$-module M as:

$$M = \bigoplus_{\tau : W \hookrightarrow \mathcal{O}} M_\tau.$$

Note that

$$\mathrm{Hom}_{\mathcal{O} \otimes_{\mathbf{Z}_p} W}(M, M') = \bigoplus_{\tau : W \hookrightarrow \mathcal{O}} \mathrm{Hom}_{\mathcal{O}}(M_\tau, M'_\tau).$$

Lemma 3.1.2.3. *If V is equipped with a Γ_K-equivariant $\mathcal{O}$-module structure then for $M^i_\tau := M_\tau \cap M^i$ we have*

$$M^i = \bigoplus_{\tau : W \hookrightarrow \mathcal{O}} M^i_\tau$$

and furthermore the σ-semilinear map $\varphi^i_M|_{M^i_\tau} : M^i_\tau \to M$ factors through $M_{\sigma\tau}$.

Note that $\sigma\tau = \tau\sigma$.

Proof. It is immediate that $M^i = \bigoplus_\tau M^i_\tau$, as the induced action of $\mathcal{O}$ on M preserves the filtration. Let $j_i : M^i_\tau \to M^i$ be the inclusion, so $j_i(wm) = \tau(w) j_i(m)$ for $w \in W$ and $m \in M^i_\tau$. The fact that φ^i_M is σ-semilinear implies that for $m \in M^i_\tau$ and $w \in W$,

$$\sigma\tau(w)\varphi^i_M(j_i(m)) = \varphi^i_M(\tau(w) j_i(m)) = \varphi^i_M(j_i(wm)).$$ □

Lemma 3.1.2.4. *The length of M as an $\mathcal{O}$-module equals the length of V as an $\mathcal{O}$-module multiplied by $[K : \mathbf{Q}_p]$.*

Proof. The length of M as a W-module equals the length of V as a $\mathbf{Z}_p$-module by Fact 3.1.1.14(4). The rest is bookkeeping. The $\mathbf{F}_p$-dimension of $\mathcal{O}/p\mathcal{O}$ is $t = [\mathcal{O}[\frac{1}{p}] : \mathbf{Q}_p]$, and the k'-dimension of $\mathcal{O}/p\mathcal{O}$ is $s = [\mathcal{O}[\frac{1}{p}] : K]$. It follows that

$$s \lg_{\mathcal{O}}(M) = \lg_W(M) = \lg_{\mathbf{Z}_p}(V) = t \lg_{\mathcal{O}}(V).$$

Hence $\lg_{\mathcal{O}}(M) = [K : \mathbf{Q}_p] \lg_{\mathcal{O}}(V)$. □

We can prove a result about freeness when V and M are R-modules for an artinian coefficient $\mathcal{O}$-algebra R with residue field k:

Lemma 3.1.2.5. *We have V is a free R-module if and only if M is a free R-module. When M is a free R-module, all of the M_τ^i are free R-direct summands. All of the M_τ has the same rank.*

Proof. A finitely generated R-module N is free if and only if $\lg_{\mathcal{O}}(N) = n \lg_{\mathcal{O}}(R)$ for $n = \dim_k N/\mathfrak{m}_R N$, as we see via Nakayama's lemma applied to a map $R^n \to N$ inducing an isomorphism modulo $\mathfrak{m}_R$. From the exact sequence of Fontaine-Laffaille modules

$$0 \to \mathfrak{m}_R M \to M \to M/\mathfrak{m}_R M \to 0$$

and the fact that T_{cris} is covariant and exact, we see that $T_{\text{cris}}(M/\mathfrak{m}_R M) = V/\mathfrak{m}_R V$. By Lemma 3.1.2.4, if $\dim_k V/\mathfrak{m}_R V = n$ then $M/\mathfrak{m}_R M$ is a k-vector space of dimension $[K : \mathbf{Q}_p]n$. Thus to relate R-freeness of M and V we just need to show that $\lg_{\mathcal{O}}(M) = [K : \mathbf{Q}_p] \lg_{\mathcal{O}}(V)$, which is Lemma 3.1.2.4.

Now suppose M is a free R-module. By functoriality, the $\mathbf{Z}_p$-module direct summands M_τ of M are each R-submodules, so each M_τ is an R-module direct summand of M. Hence each M_τ is R-free when M is free. To deduce the same for each M_τ^i, we just need that each M_τ^i is an R-module summand. By R-freeness of M, it suffices to show that each $M_\tau^i/\mathfrak{m}_R M_\tau^i \to M/\mathfrak{m}_R M$ is injective. Since M_τ^i is the "τ-component" of M^i by Lemma 3.1.2.3 it is an R-module summand of M^i. Thus it suffices to show that

$$M^i/\mathfrak{m}_R M^i \to M/\mathfrak{m}_R M$$

is injective for all i. But this follows from the abelianness in Fact 3.1.1.5.

To check that all of the M_τ have the same rank, by freeness it suffices to check that $\dim_k \overline{M}_\tau$ is independent of τ. As all $\mathbf{Z}_p$-embeddings of the *unramified* W into $\mathcal{O}$ are of the form $\sigma^i\tau$ for some fixed $\mathbf{Z}_p$-embedding τ and σ has finite order, it suffices to show that

$$\dim_k \overline{M}_\tau \geq \dim_k \overline{M}_{\sigma\tau}.$$

As each $\overline{M}_\tau^i$ is a k-module direct summand of $\overline{M}_\tau$, $\overline{M}_\tau$ is isomorphic to $\mathrm{gr}^\bullet \overline{M}_\tau$. Because $\varphi^i_{\overline{M}}(\overline{M}^{i+1}) = 0$, we obtain a map

$$\sum_i \varphi^i_{M_\tau} : \mathrm{gr}^\bullet \overline{M}_\tau \to \overline{M}_{\sigma\tau}.$$

As Fontaine-Laffaille modules satisfy

$$\overline{M} = \sum_i \varphi^i(\overline{M}^i)$$

the map $\sum_i \varphi^i_{M_\tau}$ is surjective. This completes the proof. □

Remark 3.1.2.6. We get a set of Fontaine-Laffaille weights for each $\mathbf{Z}_p$-embedding $\tau : W \hookrightarrow \mathcal{O}$. We can also define the multiplicity of a weight w_τ to be the rank of the R-module $M_\tau^{w_\tau}/M_\tau^{w_\tau+1}$. The number of Fontaine-Laffaille weights, counted with multiplicity, for each embedding is the same. We say the Fontaine-Laffaille weights with respect to an embedding are distinct if each has multiplicity 1. This is analogous to the way a Hodge-Tate representation of Γ_K over a p-adic field splitting K over $\mathbf{Q}_p$ has a set of Hodge-Tate weights for each $\mathbf{Q}_p$-embedding of K into that field.

We can now define a notion of a tensor product for Fontaine-Laffaille modules that are also R-modules for a coefficient ring R over $\mathcal{O}$.

Definition 3.1.2.7. Define $M_1 \otimes_R M_2$ to be the module $M_1 \otimes_R M_2$ together with filtration defined by $(M_1 \otimes_R M_2)^n = \sum_{i+j=n} M_1^i \otimes_R M_2^j$ and with $\varphi^n_{M_1 \otimes_R M_2}$ defined in the obvious way on the pieces.

Lemma 3.1.2.8. *Suppose that M_1, M_2, and $M_1 \otimes_R M_2$ are all in $\mathrm{MF}^{f,[a,b]}_{W,\text{tor}}$ with $0 \leq b - a \leq p - 2$. The natural map $T_{\text{cris}}(M_1) \otimes_R T_{\text{cris}}(M_2) \to T_{\text{cris}}(M_1 \otimes_R M_2)$ is an isomorphism.*

Proof. We have an exact sequence

$$0 \to J \to M_1 \otimes M_2 \to M_1 \otimes_R M_2 \to 0$$

where J is generated by the extra relations imposed by R-bilinearity (beyond W-bilinearity). For $r \in R$, define $\mu_r : M_1 \otimes M_2 \to M_1 \otimes M_2$ by

$$\mu_r(m_1 \otimes m_2) = (rm_1) \otimes m_2 - m_1 \otimes (rm_2).$$

Then $J = \sum_{r \in R} \mathrm{Im}(\mu_r)$; this is an object of $\mathrm{MF}^{f,[a,b]}_{W,\mathrm{tor}}$ by Fact 3.1.1.5. We will show that $T_{\mathrm{cris}}(J)$ is the kernel of $T_{\mathrm{cris}}(M_1 \otimes M_2) \to T_{\mathrm{cris}}(M_1) \otimes_R T_{\mathrm{cris}}(M_2)$.

It suffices to show that if N_1 and N_2 are sub-objects of $M_1 \otimes M_2$ then $T_{\mathrm{cris}}(N_1 + N_2) = T_{\mathrm{cris}}(N_1) + T_{\mathrm{cris}}(N_2)$. Indeed, granting this we would know that

$$T_{\mathrm{cris}}(J) = \sum_{r \in R} T_{\mathrm{cris}}(\mu_r).$$

But by functoriality $T_{\mathrm{cris}}(\mu_r)$ is the map $T_{\mathrm{cris}}(M_1) \otimes T_{\mathrm{cris}}(M_2) \to T_{\mathrm{cris}}(M_1) \otimes T_{\mathrm{cris}}(M_2)$ given by $v_1 \otimes v_2 \mapsto rv_1 \otimes v_2 - v_1 \otimes rv_2$, so $T_{\mathrm{cris}}(J)$ is the kernel of $T_{\mathrm{cris}}(M_1 \otimes M_2) \to T_{\mathrm{cris}}(M_1) \otimes_R T_{\mathrm{cris}}(M_2)$ as desired.

To prove that $T_{\mathrm{cris}}(N_1 + N_2) = T_{\mathrm{cris}}(N_1) + T_{\mathrm{cris}}(N_2)$, consider the exact sequence

$$0 \to N_1 \cap N_2 \to N_1 \oplus N_2 \to N_1 + N_2 \to 0.$$

As T_{cris} preserves direct sums, it suffices to show that

$$T_{\mathrm{cris}}(N_1) \cap T_{\mathrm{cris}}(N_2) = T_{\mathrm{cris}}(N_1 \cap N_2).$$

But this follows from the exactness of T_{cris} and the left exact sequence

$$0 \to N_1 \cap N_2 \to N_1 \oplus N_2 \to M_1 \otimes M_2$$

where the second map is $(n_1, n_2) \mapsto n_1 - n_2$. ☐

3.1.3 Duality

Let R be a coefficient ring over $\mathcal{O}$ and $M \in \mathrm{MF}^f_{W,\mathrm{tor}}$ have the structure of a free R-module. Fix $L \in \mathrm{MF}^f_{W,\mathrm{tor}}$ with an R-structure so that for each τ, L_τ is a free R-module of rank 1 with $L_\tau^{s_\tau} = L_\tau$ and $L_\tau^{s_\tau+1} = 0$ for some s_τ (the analogue of a character taking values in $R^\times$). We will define a dual relative to L akin to Cartier duality. This will be useful for studying pairings.

Definition 3.1.3.1. For an M as above, define $M^\vee = \mathrm{Hom}_{R \otimes_{\mathbf{Z}_p} W}(M, L)$ with a filtration given by

$$\mathrm{Fil}^i M^\vee = \{\psi \in \mathrm{Hom}_{R \otimes_{\mathbf{Z}_p} W}(M, L) : \psi(M^j) \subset L^{i+j} \text{ for all } j \in \mathbf{Z}\}.$$

For $\psi \in \mathrm{Fil}^i M^\vee$, define $\varphi^i_{M^\vee}(\psi)$ to be the unique function in $\mathrm{Hom}_{R \otimes_{\mathbf{Z}_p} W}(M, L)$ such that

$$\varphi^i_{M^\vee}(\psi)(\varphi^j_M(m)) = \varphi^{i+j}_L(\psi(m)).$$

for all $m \in M^j$ and j.

If $\varphi^i_{M^\vee}$ exists, it is unique since the images of the φ^j_M's span M additively. Likewise, if $\varphi^i_{M^\vee}$ exists for all i they are automatically σ-semilinear and satisfy $p\varphi^{i+1}_{M^\vee} = \varphi^i_{M^\vee}|_{\mathrm{Fil}^{i+1} M^\vee}$. We check $\varphi^i_{M^\vee}(\psi)$ is well-defined in the following lemma. The fact that all of the M^i_τ are free R-module direct summands of M_τ (by Lemma 3.1.2.5) is crucial.

Lemma 3.1.3.2. *The function $\varphi^i_{M^\vee}(\psi)$ is well-defined, and the filtration can equivalently be described as*

$$\mathrm{Fil}^i M^\vee = \bigoplus_{\tau : W \hookrightarrow \mathcal{O}} \mathrm{Hom}_R(M_\tau / M_\tau^{1+s_\tau-i}, L_\tau).$$

Proof. We first establish the alternate description of $\mathrm{Fil}^i M^\vee$. Because

$$\mathrm{Hom}_{R\otimes_{\mathbf{Z}_p} W}(M, L) = \bigoplus_{\tau: W \hookrightarrow \mathcal{O}} \mathrm{Hom}_R(M_\tau, L_\tau),$$

and $L_\tau^{s_\tau} = L_\tau$ while $L_\tau^{s_\tau+1} = 0$, an element $\psi_\tau \in \mathrm{Hom}_R(M_\tau, L_\tau)$ satisfies $\psi_\tau(M_\tau^j) \subset L_\tau^{i+j}$ if and only if $\psi_\tau(M_\tau^j) = 0$ whenever $i+j > s_\tau$. This says exactly that ψ_i factors through $M_\tau/M_\tau^{1+s_\tau-i}$. Because $M_\tau^{1+s_\tau-i}$ is an R-module direct summand, hence free with free complement, a morphism $M_\tau/M_\tau^{1+s_\tau-i} \to L_\tau$ is equivalent to a $\psi_\tau : M_\tau \to L_\tau$ such that $\psi_\tau(M^{1+s_\tau-i}) = 0$. Thus $\mathrm{Fil}^i M_\tau^\vee = \mathrm{Hom}_R(M_\tau/M_\tau^{1+s_\tau-i}, L_\tau)$ as desired.

We will construct $\varphi^i_{M^\vee} : \mathrm{Fil}^i M^\vee \to M^\vee$ using the exact sequence

$$0 \to \bigoplus_{r=a+1}^{b} M^r \to \bigoplus_{r=a}^{b} M^r \to M \to 0 \tag{3.1.3.1}$$

of [FL82, Lemme 1.7]. The first map sends $(m_r)_{r=a+1}^{r=b}$ to $(pm_r - m_{r+1})_{r=a}^{r=b}$ (with the convention that $m_a = 0$ and $m_{b+1} = 0$), and the second map is $\sum_{r=a}^b \varphi^i_M$. For $\psi \in \mathrm{Fil}^i M^\vee$, consider the map

$$\phi : \bigoplus_{r=a}^{b} M^r \to L_\sigma$$

induced by the $\varphi_L^{i+r} \circ \psi : M^r \to L_\sigma$. For $(m_r)_{r=a+1}^{r=b}$ in $\bigoplus_{r=a+1}^b M^r$, we compute that

$$\begin{aligned}\phi((m_r)_{r=a+1}^{r=b}) &= \sum_{j=a}^{b} \varphi_L^{i+j}(\psi((pm_j - m_{j+1}))) \\ &= \sum_{j=a}^{b} p\varphi_L^{i+j}(\psi(m_j)) - \sum_{j=a}^{b} \varphi_L^{i+j}(\psi(m_{j+1})).\end{aligned}$$

But $\varphi_L^{i+j}|_{L^{i+j+1}} = p\varphi_L^{i+j+1}$, so this difference is

$$\sum_{j=a}^{b} p\varphi_L^{i+j}(\psi(m_j)) - \sum_{j=a+1}^{b+1} p\varphi_L^{i+j}(\psi(m_j))$$

which vanishes as $m_{b+1} = 0$ and $m_a = 0$. Hence ϕ factors through the quotient M of (3.1.3.1), giving the desired well-defined map $\varphi^i_{M^\vee}$. □

Lemma 3.1.3.3. *The Fontaine-Laffaille module $M^\vee$ is an object of $\mathrm{MF}^f_{W,\mathrm{tor}}$.*

Proof. We need to show that $\sum_i \varphi^i_{M^\vee}(\mathrm{Fil}^i M^\vee) = M^\vee$. It suffices to show that the inclusion

$$\sum_i \varphi^i_{M^\vee}(\mathrm{Fil}^i M^\vee) \hookrightarrow M^\vee$$

is an equality. By Nakayama's lemma, it suffices to show that the reduction modulo $\mathfrak{m}_R$ is surjective. For an R-module N, let $\overline{N}$ denote the reduction modulo $\mathfrak{m}_R$. We may pick free R-modules N_τ^i such that $M_\tau^i = N_\tau^i \oplus M_\tau^{i+1}$ as each M_τ^i is a (free) direct summand of the R-free M_τ that is an R-free direct summand of M. Because $p \cdot \varphi_M^{i+1} = \varphi_M^i|_{M^{i+1}}$, we see $\varphi_M^i(\overline{M}_\tau^i) = \varphi_M^i(\overline{N}_\tau^i)$, so

$$\overline{M}_{\sigma\tau} = \sum_i \varphi_M^i(\overline{N}_\tau^i).$$

By dimension reasons $\varphi_M^i|_{N_\tau^i}$ is injective and the sum is direct. We also know that $\varphi_L^i|_{\overline{L_\tau}} = 0$ for $i < s_\tau$ because $p \cdot \varphi_L^{j+1} = \varphi_L^j|_{L^{j+1}}$.

As M_τ and L_τ are free R-module summands of M and L for all τ, $\overline{M^\vee} = \overline{M}^\vee$ by Lemma 3.1.3.2. We can describe an element $\psi \in \mathrm{Fil}^i\,\overline{M^\vee}$ as a collection of $\psi_{\tau,j} \in \bigoplus_{\tau,j} \mathrm{Hom}_R(\overline{N}^j_\tau, \overline{L}^{i+j}_\tau)$. But $\overline{L}^{i+j}_\tau$ is one-dimensional over k if $i+j \le s_\tau$, and is zero otherwise. Then for $f = \varphi^i_{M^\vee}(\psi)$ and $m = \sum_{\tau,j} \varphi^j_M(n_{\tau,j})$ with $n_{\tau,j} \in \overline{N}^j_\tau$, by construction we have

$$f(m) = \sum_{\tau,j} \varphi^{i+j}_L(\psi(n_{\tau,j})).$$

But $\varphi^{i+j}_L(\psi(n_{\tau,j}))$ is forced to be zero unless $i+j = s_\tau$, in which case it can take on any non-zero value in $\overline{L}_\tau$ (depending on the choice of ψ). Thus

$$\varphi^i_{M^\vee}(\mathrm{Fil}^i\,\overline{M}^\vee) = \bigoplus_\tau \mathrm{Hom}\left(\varphi^{s_\tau - i}_M(\overline{N}^{s_\tau - i}_\tau), \overline{L}_{\sigma\tau}\right).$$

Summing over i, and using the sum decomposition $\overline{M} = \sum_{\tau,i} \varphi^i_M(\overline{N}^i_\tau)$ gives that

$$\sum_i \varphi^i_{\overline{M}^\vee}(\mathrm{Fil}^i\,\overline{M}^\vee) = \mathrm{Hom}(\overline{M}, \overline{L}).$$

This shows the desired surjectivity. □

Remark 3.1.3.4. For fixed $\mathbf{Z}_p$-embedding $\tau : W \hookrightarrow \mathcal{O}$, if the Fontaine-Laffaille weights (Remark 3.1.2.6) of M with respect to τ are $\{w_{\tau,i}\}_i$ then the Fontaine-Laffaille weights of $M^\vee$ with respect to τ are $\{s_\tau - w_{\tau,i}\}_i$. This is just bookkeeping.

Letting ν be the Galois representation on the free rank-1 R-module corresponding to L, we define the *dual* $V^\vee = \mathrm{Hom}_{R[\Gamma_K]}(V, R(\nu))$ for a discrete Γ_K-representation on a finite free R-module V.

Lemma 3.1.3.5. *For a morphism $f : M \to N$ in $\mathrm{MF}^{f,[a,b]}_{W,\mathrm{tor}}$ with $b - a \le \frac{p-2}{2}$, there is a natural isomorphism $T_{\mathrm{cris}}(M^\vee) \simeq T_{\mathrm{cris}}(M)^\vee$ and $T_{\mathrm{cris}}(f^\vee) = T_{\mathrm{cris}}(f)^\vee$.*

Proof. We prove this by studying the evaluation pairing $M \otimes_R M^\vee \to L$. It is straightforward to verify that this pairing is a morphism of Fontaine-Laffaille modules. Because $b - a \le \frac{p-2}{2}$, we obtain a pairing of Galois-modules

$$T_{\mathrm{cris}}(M) \otimes_R T_{\mathrm{cris}}(M^\vee) = T_{\mathrm{cris}}(M \otimes_R M^\vee) \to T_{\mathrm{cris}}(L) = \nu_R. \tag{3.1.3.2}$$

We will now prove that this pairing is perfect when $R = k$. We will do so by inducting on the dimension of the k-vector space M. The case of dimension 0 is clear. Also, if $M \ne 0$ the pairing of Fontaine-Laffaille modules is non-zero (look at the pairing of vector spaces $M_\tau \times \mathrm{Hom}(M_\tau, L_\tau) \to L_\tau$). Thus the pairing of Galois-modules is non-zero if $M \ne 0$ as T_{cris} is faithful.

Now we use induction, so we can assume $M \ne 0$. The annihilator of $T_{\mathrm{cris}}(M^\vee)$ is $T_{\mathrm{cris}}(M_1)$ for some $f : M_1 \hookrightarrow M$ because the essential image of T_{cris} is closed under taking sub-objects. We know M_1 is a proper sub-object as the pairing is non-zero. Observe that we may define the dual $f^\vee : M^\vee \to M_1^\vee$ by precomposition: it is obviously surjective since we are over a field. For $v_1 \in T_{\mathrm{cris}}(M_1)$ and $v_2 \in T_{\mathrm{cris}}(M^\vee)$, we must have

$$0 = \langle v_1, f^\vee v_2 \rangle = \langle f(v_1), v_2 \rangle.$$

But the pairing $T_{\mathrm{cris}}(M_1) \otimes T_{\mathrm{cris}}(M_1^\vee) \to T_{\mathrm{cris}}(L)$ is non-degenerate by induction, and $f^\vee$ is surjective, so this means that $v_1 = 0$. Thus $T(M_1)$ and hence M_1 are trivial. Over the field k, this ensures the pairing is perfect.

For the general case, we shall use the basic fact that for a coefficient ring R, if N_1 and N_2 are free R-modules of the same rank with an R-bilinear pairing $N_1 \times N_2 \to R$, the pairing is perfect if the reduction (modulo $\mathfrak{m}_R$) $\overline{N_1} \times \overline{N_2} \to k$ is perfect. Apply this to $T_{\mathrm{cris}}(M) \times T_{\mathrm{cris}}(M^\vee) \to T_{\mathrm{cris}}(L)$.

The statement $T_{\mathrm{cris}}(f^\vee) = T_{\mathrm{cris}}(f)^\vee$ is just functoriality. □

3.2 Fontaine-Laffaille Deformation Condition

Let $G = \mathrm{GSp}_r$ or GO_r, and consider a representation $\overline{\rho} : \Gamma_K \to G(k)$ with similitude character $\overline{\nu}$, where $K = W[\frac{1}{p}]$ for $W = W(k')$ with finite k'. Let $\overline{V}$ be the underlying vector space for $\overline{\rho}$ using the standard representation of G. Take $\mathcal{O}$ to be the Witt vectors of k, and assume $\mathcal{O}[\frac{1}{p}]$ splits K over $\mathbf{Q}_p$. Fix a lift $\nu : \Gamma_K \to \mathcal{O}^\times$ of $\overline{\nu}$ that is crystalline with Hodge-Tate weights $\{s_\tau\}_\tau$ in an interval of length $p-2$, corresponding to a filtered Dieudonné module L.

We suppose that $\overline{\rho}$ is torsion-crystalline with Hodge-Tate weights in an interval $[a, b]$ with $0 \leq b - a \leq \frac{p-2}{2}$ so we can use Fontaine-Laffaille theory. Let $\overline{M}$ be the corresponding Fontaine-Laffaille module (using Fact 3.1.1.14(6)), with Fontaine-Laffaille weights $\{w_{\tau,i}\}_{\tau,i}$. In this section we define and study the Fontaine-Laffaille deformation condition assuming that for each $\mathbf{Z}_p$-embedding $\tau : W \hookrightarrow \mathcal{O}$ the Fontaine-Laffaille weights are multiplicity-free as in Remark 3.1.2.6 (rank 1 jumps in the filtration).

3.2.1 Definitions and Basic Properties

As $\overline{V}$ is an k-linear representation of Γ_K, $\overline{M}$ becomes a $k' \otimes_{\mathbf{F}_p} k$-module and in particular a k-vector space.

Definition 3.2.1.1. For a coefficient ring R over $\mathcal{O} = W(k)$, define $D^{\mathrm{FL}}_{\overline{\rho}}(R)$ to be the collection of lifts $\rho : \Gamma_K \to G(R)$ of $\overline{\rho}$ with similitude character ν_R that lie in the essential image of T_{cris} restricted to the full subcategory $\mathrm{MF}^{f,[a,b]}_{W,\mathrm{tor}}$. Such a deformation is called a *Fontaine-Laffaille deformation.*

We will analyze this deformation condition when the Fontaine-Laffaille weights of $\overline{\rho}$ are multiplicity-free for each fixed embedding $\tau : W \hookrightarrow \mathcal{O}$ (when the jumps in the filtration of each $\overline{M}_\tau$ are 1-dimensional over k).

Theorem 3.2.1.2. *Under the multiplicity-free assumption on Fontaine-Laffaille weights, $D^{\mathrm{FL}}_{\overline{\rho}}$ is liftable. If $\mathfrak{u}$ is the Lie algebra of the unipotent radical of a Borel subgroup of G, the dimension of the tangent space of $D^{\mathrm{FL}}_{\overline{\rho}}$ is*

$$[K : \mathbf{Q}_p] \dim_k \mathfrak{u} + H^0(\Gamma_K, \mathrm{ad}^0(\overline{\rho})).$$

If $\rho : \Gamma_K \to G(\mathcal{O})$ is an inverse limit of Fontaine-Laffaille deformations of $\overline{\rho}$ to $\mathcal{O}/p^n\mathcal{O}$ for all $n \geq 1$, it is a lattice in a crystalline representation with the same Fontaine-Laffaille weights as $\overline{\rho}$.

The proof of this theorem will occur over the remainder of §3.2. The key pieces are Proposition 3.2.1.7, Proposition 3.2.2.1, and Proposition 3.2.3.1.

To understand $D^{\mathrm{FL}}_{\overline{\rho}}$, we must express the orthogonal or symplectic pairing in the language of Fontaine-Laffaille modules. For a Galois module V which is a free R-module, recall we defined $V^\vee = \mathrm{Hom}_{R[\Gamma_K]}(V, \nu_R)$. For a deformation of $\overline{\rho}$ to a coefficient ring R, we obtain an $R[\Gamma_K]$-module V together with an isomorphism $\eta : V \simeq V^\vee$ coming from the pairing. Let $\epsilon = 1$ for $G = \mathrm{GO}_r$ and $\epsilon = -1$ for $G = \mathrm{GSp}_r$. The fact that $\langle v, w \rangle = \epsilon \langle w, v \rangle$ is equivalent to $\eta^* = \epsilon\eta$, where η^* is the map $V \simeq V^{\vee\vee} \to V^\vee$ induced by double duality.

There is a corresponding Fontaine-Laffaille module M such that $T_{\mathrm{cris}}(M) = V$, and M is a free R-module by Lemma 3.1.2.5.

Lemma 3.2.1.3. *For a coefficient ring R, suppose V is a lift of $\overline{V}$ as an $R[\Gamma_K]$-module that is finite free over R, corresponding to a Fontaine-Laffaille module M. An isomorphism of $R[\Gamma_K]$-modules*

$$\eta : V \simeq V^\vee$$

such that $\eta(v)(w) = \epsilon\eta(w)(v)$ is equivalent to an R-linear isomorphism of Fontaine-Laffaille modules

$$\gamma : M \simeq M^\vee$$

such that $\gamma(m)(n) = \epsilon\gamma(n)(m)$.

Proof. As the Hodge-Tate weights of $\overline{\rho}$ lie in an interval of length $\frac{p-2}{2}$, Lemma 3.1.2.8 and Lemma 3.1.3.5 hold. In particular, $T_{\mathrm{cris}}(M^\vee) = T_{\mathrm{cris}}(M)^\vee$. As T_{cris} is fully faithful in this range, we see that a map η is equivalent to a map γ, and one is an isomorphism if and only if the other one is. It remains to check that γ is symmetric or alternating if and only if η is. Let η^* and γ^* denote the isomorphisms respectively given by

$$V \simeq V^{\vee\vee} \xrightarrow{\eta^\vee} V^\vee \quad \text{and} \quad M \simeq M^{\vee\vee} \xrightarrow{\gamma^\vee} M^\vee.$$

A straightforward check shows that T_{cris} carries η^* to γ^*, and hence $\eta = \epsilon\eta^*$ if and only if $\gamma = \epsilon\gamma^*$. ☐

Lemma 3.2.1.4. *An R-linear isomorphism of Fontaine-Laffaille modules $\gamma : M \simeq M^\vee$ for which $\gamma(m)(n) = \epsilon\gamma(n)(m)$ is equivalent to a perfect ϵ-symmetric R-bilinear pairing $\langle\cdot,\cdot\rangle : M \times M \to L_R$ satisfying*

- $\langle M^i, M^j\rangle \subset L^{i+j}$;
- $\langle \varphi^i_M(m), \varphi^j_M(n)\rangle = \varphi^{i+j}_L\langle m, n\rangle$.

Proof. This is just writing out what $\gamma : M \to M^\vee$ being a morphism of Fontaine-Laffaille modules means for the pairing $\langle m, n\rangle = \gamma(m)(n)$.

For γ to preserve the filtration says exactly that

$$\gamma(M^i) \subset \mathrm{Fil}^i M^\vee = \{\psi \in \mathrm{Hom}_R(M, L) : \psi(M^j) \subset L^{i+j}\}.$$

This is equivalent to $\langle M^i, M^j\rangle \subset L^{i+j}$ for all i, j. The compatibility of γ with the φ's says exactly that for $m \in M^i$

$$\varphi^i_{M^\vee}(\gamma(m)) = \gamma(\varphi^i_M(m)).$$

Evaluating on any $\varphi^j_M(n) \in M$ and using the definition of $M^\vee$ we see

$$\varphi^i_{M^\vee}(\gamma(m))(\varphi^j_M(n)) = \varphi^{i+j}_L(\gamma(m)(n)) = \varphi^{i+j}_L(\langle m, n\rangle).$$

Evaluating $\gamma(\varphi^i_M(m))$, we see that

$$\gamma(\varphi^i_M(m))(\varphi^j_M(n)) = \langle\varphi^i_M(m), \varphi^j_M(n)\rangle.$$

Thus, γ being compatible with φ's is equivalent to $\langle\varphi^i_M(m), \varphi^j_M(n)\rangle = \varphi^{i+j}_L(\langle m, n\rangle)$. ☐

In particular, the pairing $\overline{V} \times \overline{V} \to \overline{\nu}$ gives a perfect pairing $\langle\cdot,\cdot\rangle_{\overline{M}} : \overline{M} \times \overline{M} \to \overline{L}$.

Corollary 3.2.1.5. *For a coefficient ring R, a lift $\rho \in D^{\mathrm{FL}}_{\overline{\rho}}(R)$ is equivalent to a Fontaine-Laffaille module $M \in \mathrm{MF}^{f,[a,b]}_{W,\mathrm{tor}}$ that is free as an R-module for which there exists a perfect ϵ-symmetric R-bilinear pairing $\langle\cdot,\cdot\rangle : M \times M \to L_R$ satisfying*

- $\langle M^i, M^j\rangle \subset L^{i+j}$;
- $\langle\varphi^i_M(m), \varphi^j_M(n)\rangle = \varphi^{i+j}_L\langle m, n\rangle$.

such that $(M, \langle\cdot,\cdot\rangle)$ reduces to $(\overline{M}, \langle\cdot,\cdot\rangle_{\overline{M}})$.

Proof. This follows by combining the two previous lemmas. Note that the pairing $\langle\cdot,\cdot\rangle$ is automatically perfect as it lifts the perfect pairing $\langle\cdot,\cdot\rangle_{\overline{M}}$. ☐

Corollary 3.2.1.6. *$D^{\mathrm{FL}}_{\overline{\rho}}$ is a deformation condition.*

Proof. This follows from the fact that for a morphism of coefficient rings $R \to R'$, $R' \otimes_R T_{\mathrm{cris}}(M) = T_{\mathrm{cris}}(R' \otimes_R M)$, exactness properties of T_{cris} on $\mathrm{MF}^f_{W,\mathrm{tor}}$, and Corollary 3.2.1.5. For example, to check that $D^{\mathrm{FL}}_{\overline{\rho}}$ is a sub-functor of $\mathcal{D}_{\overline{\rho}}$, let R be a coefficient ring and M be the Fontaine-Laffaille module corresponding to $\rho \in D^{\mathrm{FL}}_{\overline{\rho}}(R)$. Then $R' \otimes_R T_{\mathrm{cris}}(M))$ lies in the essential image of T_{cris}, and $R' \otimes_R M$ admits a perfect ϵ-symmetric R'-bilinear pairing as in Corollary 3.2.1.5 given by extending the pairing on M. This shows that $\rho_{R'} \in D^{\mathrm{FL}}_{\overline{\rho}}(R')$. A similar argument checks Definition 2.2.2.7(2). ☐

It is simple to understand characteristic-zero points of the deformation functor.

Proposition 3.2.1.7. *Suppose we are given a compatible collection of Fontaine-Laffaille deformations $\rho_i : \Gamma_K \to G(R_i)$, where $\{R_i\}$ is a co-final system of artinian quotients of the valuation ring R of a finite extension of $\mathcal{O}[\frac{1}{p}]$ with the same residue field as $\mathcal{O}$. Then $\rho = \varprojlim \rho_i$ is crystalline (more precisely, a lattice in a crystalline representation) with indexed tuple of Hodge-Tate weights equal to the corresponding indexed-tuple of Fontaine-Laffaille weights of $\overline{\rho}$.*

Proof. It is straightforward to verify that the inverse limit of the Fontaine-Laffaille modules corresponding to ρ_i is in $\mathcal{D}_K$. Then the Proposition follows from combining Fact 3.1.1.14(2) and (5). Our convention that the cyclotomic character has Hodge-Tate weight -1 makes the Hodge-Tate weights and Fontaine-Laffaille weights match (Remark 3.1.1.17). □

3.2.2 Liftability

In this section, we analyze liftability by constructing lifts of Fontaine-Laffaille modules. Lifting the underlying module, filtration, and pairing will be relatively easy. Constructing lifts of the φ_M^i compatible with these choices requires substantial work. Let $\mathcal{W}_{\mathrm{FL},\tau}$ denote the Fontaine-Laffaille weights of $\overline{\rho}$ with respect to a $\mathbf{Z}_p$-embedding $\tau : W \hookrightarrow \mathcal{O}$, corresponding to the jumps in the filtration of $\overline{M}_\tau$.

Proposition 3.2.2.1. *Under the assumption that the Fontaine-Laffaille weights lie in an interval of length $\frac{p-2}{2}$ and are multiplicity-free for each $\tau : W \hookrightarrow \mathcal{O}$, the deformation condition $D_{\overline{\rho}}^{\mathrm{FL}}$ is liftable.*

Let $\rho : \Gamma_K \to G(R)$ be a Fontaine-Laffaille deformation of $\overline{\rho}$. Let M and $\overline{M}$ be the corresponding Fontaine-Laffaille modules for ρ and $\overline{\rho}$, which decompose as

$$M = \bigoplus_\tau M_\tau \quad \text{and} \quad \overline{M} = \bigoplus_\tau \overline{M}_\tau.$$

Each M_τ is a free R-module by Lemma 3.1.2.5. Recall that there is a filtration $\{M_\tau^i\}$ on M_τ given by R-module direct summands, and that $\varphi_M^i(M_\tau^i) \subset M_{\sigma\tau}$. In particular, there exist free rank-1 R-modules $N_\tau^i \subset M_\tau^i$ such that $M_\tau^i = N_\tau^i \oplus M_\tau^{i+1}$. As the pairing is $\mathcal{O}$-bilinear, the pairings $M_\tau \times M_\tau \to L_\tau$ are collectively equivalent to the pairing $M \times M \to L$, so to lift the pairing and check compatibility it suffices to do so on M_τ. To analyze liftability, we may work with each M_τ separately (since $R \otimes_{\mathbf{Z}_p} W = \prod_\tau R_\tau$ with τ varying through $\mathbf{Z}_p$-embeddings $W \hookrightarrow \mathcal{O} \to R$).

By a *basis* for M_τ, we mean a basis for it as an R-module. By Lemma 3.1.2.5, the rank of M_τ is r. For $G = \mathrm{GSp}_r$ with r even, the *standard alternating pairing* with respect to a chosen basis is the one given by the block matrix

$$\begin{pmatrix} 0 & I'_{r/2} \\ -I'_{r/2} & 0 \end{pmatrix}$$

where I'_m denotes the anti-diagonal matrix with 1's on the diagonal. For $G = \mathrm{GO}_r$, the *standard symmetric pairing* with respect to the basis is the one given by the matrix I'_r.

Example 3.2.2.2. Take $R = k$ and fix an embedding $\tau : W \hookrightarrow \mathcal{O}$. Let $w_1, \dots, w_r$ be the Fontaine-Laffaille weights of M_τ, and recall that $w_i + w_{r+1-i} = s_\tau$ because $M \simeq M^\vee$. Pick $v_i \in M_\tau^{w_i} - M_\tau^{w_i+1}$. Since $\varphi_M^i|_{M_{i+1}} = p\varphi_M^{i+1} = 0$,

$$M_{\sigma\tau} = \sum_i \varphi^i(M_\tau^i) = \mathrm{span}_k\, \varphi_M^{w_i}(v_i).$$

Note that $\{\varphi_M^{w_i}(v_i)\}$ is a k-basis for $M_{\sigma\tau}$, as the left side has k-dimension r and there are r Fontaine-Laffaille weights for $\sigma\tau$. Furthermore, compatibility with the pairing means that

$$\langle \varphi_M^{w_i}(v_i), \varphi_M^{w_j}(v_j) \rangle = \varphi_L^{w_i+w_j}(\langle v_i, v_j \rangle).$$

But $\varphi_L^h|_{L_\tau} = 0$ unless $h = s_\tau$: for $h > s_\tau$ this is because $L_\tau^h = 0$, while for $h < s_\tau$ this is because $L_\tau^h = L_\tau^{h+1} = L_\tau$ and $\varphi_L^h|_{L_\tau^{h+1}} = p\varphi_L^{h+1} = 0$. Thus $\langle \varphi_M^{w_i}(v_i), \varphi_M^{w_j}(v_j) \rangle = 0$ unless $w_i + w_j = s_\tau$, in which case the pairing must be non-zero as it is perfect. If $i \neq j$, by rescaling v_i we may arrange for $\langle \varphi_M^{w_i}(v_i), \varphi_M^{w_j}(v_j) \rangle$ to be an arbitrary unit. For $G = \mathrm{GSp}_r$ or $G = \mathrm{GO}_r$ with r even this means after rescaling the pairing may be taken to be standard with respect to the basis $n_i = \varphi^{w_i}(v_i)$ of $M_{\sigma\tau}$. For $G = \mathrm{GO}_r$ with r odd and $i = [r/2]+1$, defining $c_\tau := \langle \varphi^{w_i}(v_i), \varphi^{w_i}(v_i)\rangle \in k^\times$ and rescaling $v_1, \ldots, v_{i-1}$ then brings us to the case that the pairing is c_τ times the standard pairing with respect to the basis $n_i = \varphi^{w_i}(v_i)$ of $M_{\sigma\tau}$.

Remark 3.2.2.3. The constant c_τ depends on the choice of basis $\{v_i\}$ for M_τ, so in particular is not independent of τ. This will not cause problems in later arguments.

Remark 3.2.2.4. There is a lot of notation in the following arguments. With τ fixed, we will use v_i to denote elements of $M_\tau^{w_i}$, and m_i to denote elements of $M_{\sigma\tau}$. Usually we will have $\varphi_M^{w_i}(v_i) = m_i$. If we want to index by Fontaine-Laffaille weights instead of the integers $\{1, 2, \ldots, r\}$, we will use $v'_{w_i} := v_i$ and $m'_{w_i} := m_i$.

Lemma 3.2.2.5. *Let $w_1 < w_2 < \ldots < w_r$ denote the Fontaine-Laffaille weights of M with respect to τ. There exists an R-basis $m_1 \ldots, m_r$ of $M_{\sigma\tau}$ such that $m_i = \varphi_M^{w_i}(v_i)$ where v_i is an R-basis for a complement to $M_\tau^{w_i+1}$ in $M_\tau^{w_i}$ and such that the pairing $\langle\cdot,\cdot\rangle$ on $M_{\sigma\tau}$ is an $R^\times$-multiple of the standard pairing with respect to the basis $\{m_i\}$.*

Proof. Example 3.2.2.2 shows that such a basis $\overline{v}_i$ exists over $R/\mathfrak{m}_R$: pick a lift $v_i \in N_\tau^i$ of $\overline{v}_i$, and define $m_i = \varphi_M^{w_i}(v_i)$. We know that

$$\langle \varphi_M^{w_i} v_i, \varphi_M^{w_j} v_j\rangle = \varphi_L^{w_i+w_j}(\langle v_i, v_j\rangle).$$

If $w_i + w_j > s_\tau$, this is zero because $L_\tau^{s_\tau+1} = 0$. If $w_i + w_j < s_\tau$, this is not a unit as $\varphi_L^{w_i+w_j}|_{L_\tau^{s_\tau}} = p^{s_\tau - w_i - w_j}\varphi_L^{s_\tau}$. If $w_i + w_j = s_\tau$ (equivalently, $i + j = r+1$), it is a unit of R as the pairing is perfect.

We will modify the lifts v_i and then $m_i = \varphi_M^{w_i}(v_i)$ accordingly. For $0 \le j \le r/2$ (so $j < r+1-j$), we will inductively arrange that:

1. for $i \le j$, $\langle m_i, m_h\rangle = 0$ for $h \ne r+1-i$;
2. v_i is an R-basis for a complement to $M_\tau^{w_i+1}$ in $M_\tau^{w_i}$;
3. $\langle m_i, m_{r+1-i}\rangle$ is a unit for all $1 \le i \le r$.

For $j = 0$, the first condition is vacuous and the other two conditions hold by our choice of lift. Given that these conditions hold for $j-1$ with $1 \le j \le r$, we will show how to modify the v_i so that these conditions hold for j. Let $c = \langle m_j, m_{r+1-j}\rangle \in R^\times$. For $j < h < r+1-j$, define

$$\widetilde{v}_h := v_h - \langle m_j, m_h\rangle c^{-1} v_{r+1-j}.$$

As $j \ne r+1-h$, $\langle m_j, m_h\rangle \in \mathfrak{m}_R$. We compute that

$$\langle m_j, \varphi_M^{w_h}\widetilde{v}_h\rangle = \langle m_j, m_h\rangle - \langle m_j, m_h\rangle c^{-1}\langle m_j, m_{r+1-j}\rangle = 0.$$

For $i < j$, as $r+1-i \ne h, r+1-h$ we know m_i is orthogonal to both m_h and m_{r+1-h} by the inductive hypothesis and hence $\langle m_i, \varphi_M^{w_h}\widetilde{v}_h\rangle = 0$. Thus (1) holds for the R-basis $v_1, \ldots, v_j, \widetilde{v}_{j+1}, \ldots, \widetilde{v}_{r-j}, v_{r-j+1}, \ldots, v_r$.

As $\widetilde{v}_h - v_h \in M_\tau^{w_{r+1-j}}$, $\widetilde{v}_h$ is still an R-basis for a complement to $M_\tau^{w_h+1}$ in $M_\tau^{w_h}$ (since $w_{r+1-j} > w_h$ as $h < r+1-j$), so (2) holds for this new R-basis of M_τ. Furthermore, we see that

$$\langle \varphi_M^{w_h}\widetilde{v}_h, \varphi_M^{w_{r+1-h}}\widetilde{v}_{r+1-h}\rangle - \langle m_h, m_{r+1-h}\rangle \in \mathfrak{m}_R.$$

As $\langle m_h, m_{r+1-h}\rangle$ is a unit, $\langle \varphi_M^{w_h}\widetilde{v}_h, \varphi_M^{w_{r+1-h}}\widetilde{v}_{r+1-h}\rangle$ is a unit and (3) holds. Thus we may modify the lifts v_i and then accordingly m_i to satisfy the inductive hypothesis.

Take such a basis for $j = [r/2]$. By (1),

$$\langle m_i, m_{i'}\rangle = 0$$

if $i + i' \ne r+1$ and one of i or i' is at most $r/2$. Otherwise $i' > r+1-i$ so $w_i + w_{i'} > s_\tau$ and hence the pairing is zero automatically. If r is even, rescale $v_1, \ldots, v_{r/2}$ so that $\langle m_i, m_{r+1-i}\rangle = 1$ for $i \le r/2$ using (3). If r is odd (so $G = \mathrm{GO}_r$), let $c_\tau = \langle v_{[r/2]+1}, v_{[r/2]+1}\rangle \in R^\times$ and rescale $v_1, \ldots, v_{[r/2]}$ so that $\langle m_i, m_{r+1-i}\rangle = c_\tau$ for $1 \le i \le [r/2]$. In these cases, the pairing with respect to the basis $v_1, \ldots, v_r$ is c_τ times the standard pairing. ∎

Remark 3.2.2.6. When r is odd (so $G = \mathrm{GO}_r$), to choose a basis where the pairing is standard we would need to rescale $v_{[r/2]+1}$ by a square root of the unit $\langle m_{[r/2]+1}, m_{[r/2]+1}\rangle$. This might not exist in R. Also note that the orthogonal similitude group GO_r is unaffected by a unit scaling of the quadratic form.

Now we begin the proof of Proposition 3.2.2.1. Let $R' \twoheadrightarrow R$ be a small surjection with kernel I. To lift ρ to $\rho' : \Gamma_K \to G(R')$, we can reduce to the case when I is killed by $\mathfrak{m}_R$ and $\dim_k I = 1$. Lift the R-module M_τ together with its pairing $\langle\cdot,\cdot\rangle$ over R' as follows. Choose the basis $\{m_i\}$ provided by Lemma 3.2.2.5, with respect to which $\langle\cdot,\cdot\rangle$ is ω_τ times the standard pairing for some $\omega_\tau \in R^\times$. We take $M'_{\sigma\tau}$ to be a free R'-module with basis $\{n_i\}$ reducing to the basis $\{m_i\}$ of $M_{\sigma\tau}$. Lift ω_τ to some $\omega'_\tau \in (R')^\times$ and define a pairing on M'_τ to be ω'_τ times the standard pairing on M'_τ with respect to $\{n_i\}$. Pick a lift $u_i \in M'_\tau$ of v_i, and define a filtration on M'_τ by

$$(M'_\tau)^j = \mathrm{span}_{R'}(u_i : w_i \geq j).$$

We define the module $M' = \bigoplus_{\tau:W\hookrightarrow\mathcal{O}} M'_\tau$ over $W \otimes_{\mathbf{Z}_p} R$ with filtration $(M')^i = \bigoplus_{\tau:W\hookrightarrow\mathcal{O}}(M'_\tau)^i$. It is clear the filtration reduces to the filtration on M_τ. Furthermore, the pairing $M'_\tau \times M'_\tau \to L_\tau$ with respect to $\{n_i\}$ is the standard one.

It remains to produce $\varphi^i_{M'}$ lifting φ^i_M. As always, it suffices to lift all of the $\varphi^i_{M_\tau} : M^i_\tau \to M_{\sigma\tau}$ separately. We note that the $\varphi^j_{M'_\tau} : M'^j_\tau \to M'_{\sigma\tau}$ are determined by the function $\varphi^{j+1}_{M'_\tau}$ on M'^{j+1}_τ (via the relation $p\varphi^{j+1}_{M'_\tau} = \varphi^j_{M'_\tau}|_{M'^{w_j+1}_\tau}$) together with the values $\varphi^{w_i}_{M'_\tau}(u_i)$ for $w_i \in \mathcal{W}_{\mathrm{FL},\tau}$. We will define $\varphi^{w_i}_{M'_\tau}(u_i)$ for each $w_i \in \mathcal{W}_{\mathrm{FL},\tau}$ to obtain the desired set of maps $\varphi^j_{M'} : M'^j \to M'$.

It will be more convenient to index via weights, so let $n'_{w_i} = n_i$ and $u'_{w_i} = u_i$. Let us consider defining

$$\varphi^w_{M'_\tau}(u'_w) = \sum_{i\in\mathcal{W}_{\mathrm{FL},\sigma\tau}} c_{iw} n'_i := x_w$$

for c_{iw} to be determined with the obvious restriction that c_{iw} must lift the corresponding coefficient for $\varphi^w_M(v'_w)$. We will study for which choices of $\{c_{iw}\}$ these maps are compatible with the pairing.

Lemma 3.2.2.7. *For any choice of $\{c_{iw}\}$, the elements x_w form a basis for $M'_{\sigma\tau}$.*

Proof. Note that the Fontaine-Laffaille weights of $\overline{M}$, M, and M' are the same. Consider the map

$$\sum_{i\in\mathcal{W}_{\mathrm{FL},\tau}} \varphi^i_{M'_\tau} : M'^i_\tau \to M'_{\sigma\tau}.$$

Quotienting by the maximal ideal of R', as φ'^w_M is a lift of $\varphi^w_{\overline{M}}$ we obtain a surjection

$$\sum_{i\in\mathcal{W}_{\mathrm{FL},\tau}} \varphi^i_{\overline{M}} : \overline{M}^i_\tau \twoheadrightarrow \overline{M}_{\sigma\tau}$$

as $\overline{M}_{\sigma\tau} = \sum_i \varphi^i_{\overline{M}_\tau}(\overline{M}^i_\tau)$. By Nakayama's lemma, the original map is also a surjection. Thus $\{x_w\}$ spans the free R-module $M'_{\sigma\tau}$. But $\#\{x_w\} = \mathrm{rk}_{R'}(M'_{\sigma\tau}) = r$, so $\{x_w\}$ is a basis for $M'_{\sigma\tau}$. ☐

The compatibility condition with the pairing is that

$$\langle\varphi^i_{M'_\tau}(x), \varphi^j_{M'_\tau}(y)\rangle = \varphi^{i+j}_{L_\tau}(\langle x, y\rangle).$$

Let $\epsilon = 1$ for GO_r and $\epsilon = -1$ for GSp_r with even r. Recall that for a Fontaine-Laffaille weight $i \in \mathcal{W}_{\mathrm{FL}}\tau$, we define i^* to satisfy $i + i^* = s_\tau$, so n'_i and n'_{i^*} pair non-trivially. By linearity and the relations $p\varphi^{w+1}_{M'} = \varphi^w_{M'}|_{(M')^{w+1}}$ and $\langle x, y\rangle = \epsilon\langle y, x\rangle$, it suffices to check compatibility with the pairing only when $i, j \in \mathcal{W}_{\mathrm{FL}}$, $x = n'_i$ and $y = n'_j$ and $i < j$ or $i = j = i^*$ (provided we have arranged that $p\varphi^{w+1}_{M'} = \varphi^w_{M'}|_{M'^{w+1}}$).

Remark 3.2.2.8. The case $i = j = i^*$ only occurs when the pairing is orthogonal and r is odd, for the weight of the unique basis vector which pairs with itself giving a unit.

Of course, there is no reason to expect our initial arbitrary choice of $\{c_{iw}\}$ to work. Any other choice is of the form $\{c_{iw}+\delta_{iw}\}$ where $\delta_{i,w} \in I$. The compatibility condition on M'_τ becomes

$$\sum_{w,w' \in \mathcal{W}_{\mathrm{FL},\tau}} (c_{iw}+\delta_{iw})(c_{jw'}+\delta_{jw'})\langle n'_w, n'_{w'}\rangle = \varphi^{i+j}_{L_\tau}\left(\langle n'_i, n'_j\rangle\right).$$

Expanding and using the fact that $I^2=0$, we see that we wish to choose $\{\delta_{iw}\}$ so that

$$\sum_{w,w' \in \mathcal{W}_{\mathrm{FL},\tau}} (c_{iw}\delta_{jw'}+c_{jw'}\delta_{iw})\langle n'_w, n'_{w'}\rangle = \omega'_\tau C_{ij}$$

where the constant $C_{ij} := (\omega'_\tau)^{-1}\left(\varphi^{i+j}_{L_\tau}(n'_i, n'_j) - \sum_{w,w' \in \mathcal{W}_{\mathrm{FL},\tau}} c_{iw}c_{jw'}\langle n'_w, n'_{w'}\rangle\right)$ lies in I as φ^i_M is compatible with the pairing.

Now we can simplify based on the explicit form of the pairing with respect to the basis $\{n'_w\}$. As n'_w pairs non-trivially with n'_{w^*}, for $i<j$ or $i=j=i^*$ we obtain the relation

$$\sum_{w \le w^*} (c_{iw}\delta_{jw^*}+c_{jw^*}\delta_{iw}) + \epsilon \sum_{w > w^*} (c_{iw}\delta_{jw^*}+c_{jw^*}\delta_{iw}) = C_{ij}. \tag{3.2.2.1}$$

To show that this system of linear equations has a solution, we shall interpret it as a linear transformation.

It is now convenient to index using $\{1,2,3\dots,r\}$. Recall that the Fontaine-Laffaille weights of M_τ are denoted $w_1 < w_2 < \dots < w_r$. Let $U = I^{\oplus r^2}$, and decompose U as $\bigoplus_{i=1}^r U_i$, where the coordinates of $U_i = I^{\oplus r}$ are denoted$\{\delta_{w_i,w_j}\}_{j=1}^r$. Let $U' = I^{\oplus \frac{r(r-1)}{2}+\sigma_r}$, where $\sigma_r = 1$ if there is a $w \in \mathcal{W}_{\mathrm{FL},\tau}$ for which $w=w^*$ and 0 otherwise. (So σ_r is zero unless $G = \mathrm{GO}_r$ and r is odd.) We may write $U' = \bigoplus_{i=1}^{r-r} U'_i$, where the coordinates of $U'_i = I^{\oplus r-i}$ are denoted $\{C_{w_iw_j}\}_{j=i+1}^r$, except if $\sigma_r=1$ and $w_i = w_i^*$. In that case, instead take $U'_i = I^{\oplus r-i+1}$ with coordinates denoted $\{C_{w_iw_j}\}_{j=i}^r$.

Consider the function $T: U \to U'$ given by

$$T\left((\delta_{w_i,w_h})_{ih}\right) = \left(\sum_{w_h \le w_h^*}\left(c_{w_iw_h}\delta_{w_jw_h^*} + c_{w_jw_h^*}\delta_{w_iw_h}\right) + \epsilon \sum_{w_h > w_h^*}\left(c_{w_iw_h}\delta_{w_jw_h^*} + c_{w_jw_h^*}\delta_{w_iw_h}\right)\right)_{ij}$$

where the $c_{ww'} \in R'$ matter only through their images in k since $\mathfrak{m}_R I = 0$. It suffices to show that T is surjective. As we arranged for I to be 1-dimensional over $R/\mathfrak{m}_R = k$, this is question of linear algebra over k upon fixing a k-basis of I.

We will studying particular k-linear maps $U_i \to U'_i$. To simplify notation, let $\epsilon_i = 1$ if $w_i \le w_i^*$ and $\epsilon_i = -1$ otherwise.

Lemma 3.2.2.9. *Suppose $w_i \ne w_i^*$. The linear transformation $T_i : U_i \to U'_i$ defined on*

$$(\delta_{w_iw_h})_h \mapsto (C_{w_iw_j} = \sum_{h=1}^r \epsilon_h c_{w_jw_h^*}\delta_{w_iw_h})_j$$

is surjective. It is the composition $U_i \to U \xrightarrow{T} U' \to U'_i$.

Proof. As I is one-dimensional over $R/\mathfrak{m}_R = k$, it suffices to study the matrix for this linear transformation with respect to a fixed k-basis of I. Fix $w_{h'} \in \mathcal{W}_{\mathrm{FL},\tau}$. If we take $\delta_{w_iw_h} = 0$ for $w_h \ne w_{h'}$ and $\delta_{w_iw_{h'}} = 1$, the image of $\{\delta_{w_iw_h}\}_h \in U_i$ under T_i has coordinates $C_{w_iw_j} = \epsilon_{w_{h'}}c_{w_jw^*_{h'}}$. Thus the matrix for T_i is

$$\begin{pmatrix} \epsilon_1 c_{w_{i+1}w_1^*} & \epsilon_2 c_{w_{i+1}w_2^*} & \dots & \epsilon_r c_{w_{i+1}w_r^*} \\ \epsilon_1 c_{w_{i+2}w_1^*} & \epsilon_2 c_{w_{i+2}w_2^*} & \dots & \epsilon c_{w_{i+1}w_r^*} \\ \dots & \dots & \dots & \dots \\ \epsilon_1 c_{w_rw_1^*} & \epsilon_2 c_{w_rw_2^*} & \dots & \epsilon_r c_{w_rw_r^*} \end{pmatrix}.$$

Multiplying the columns where $w_i > w_{i^*}$ by -1, the columns of this matrix are exactly the coordinates of x_{w_j} with respect to the basis $\{n'_w\}_{w \in \mathcal{W}_{\mathrm{FL},\sigma\tau}}$ as in Lemma 3.2.2.7 except that the first i rows are removed. As the $\{x_w\}$ form a basis, the columns of this matrix span U'_i.

The last statement follows from the definition. $\square$

Remark 3.2.2.10. The statement for $w_i = w_i^*$ is similar. In that case, we must have $\epsilon = 1$, and we have

$$C_{w_i w_i} = 2 \sum_j c_{w_i w_j^*} \delta_{w_i w_j}.$$

Extending the definition of T_i in Lemma 3.2.2.9, we again see that the columns of the matrix representing this transformation are truncated versions of the coordinates of x_{w_j} with some signs changed and one coordinate multiplied by 2. The image of a basis under the transformation multiplying one coordinate by 2 is still a basis, so again T_i is surjective.

Lemma 3.2.2.11. *The composition $T_{ij} : U_i \to U \xrightarrow{T} U' \to U'_j$ is zero whenever $i < j$.*

Informally, this is saying that T is block lower-triangular with diagonal blocks that are surjective.

Proof. The coordinates of U_i are $\delta_{w_i w_h}$. The coordinates of U'_j are $C_{w_j w_h}$ for $j < h$ (or $j \leq h$ if $w_j = w_j^*$). The formulas for the $C_{w_j w_h}$ in the definition of T depend only on w_j and w_h. In particular, as $i < j \leq h$, these coordinates are always zero on the image of the inclusion $U_i \to U$. □

Corollary 3.2.2.12. *T is surjective.*

Proof. The composition of $U_i \to U \to U' \to U'_i$ is exactly T_i, hence surjective. For $v \in U'$, by descending induction on i, we will construct $u_i \in U_i$ so that

$$T(u_i + \ldots + u_r) - v \in U'_1 \oplus \ldots \oplus U'_{i-1}$$

(meaning $T(u_1 + \ldots + u_r) = v$ when $i = 1$). For $i = r$, take u_r be a preimage under T_r of the component of v in U'_r. Now suppose we have selected $u_{i+1}, \ldots u_r$. Pick a preimage $u_i \in U_i$ of the projection of $T(u_{i+1} + \ldots u_r) - v$ to U'_i using the surjectivity of T_i. We know that $T_{ij}(u_i) = 0$ for $j > i$, so

$$T(u_i + \ldots + u_r) - v \in U_1 \oplus \ldots \oplus U_{i-1}.$$

For $i = 1$, we have $T(u_1 + \ldots + u_r) = v$ as desired. □

By Corollary 3.2.2.12, we can choose the $\{\delta_{ih}\}$ so that the compatibility relations (3.2.2.1) are satisfied. This defines $\varphi^w_{M'_\tau}(n'_w)$, and hence we can extend to a map $\varphi^i_{M'} : M' \to M'$ compatible with the pairing. We can finish the proof of Proposition 3.2.2.1 as follows.

Given the deformation ρ to a coefficient ring R with associated Fontaine-Laffaille module

$$M = \bigoplus_{\tau : W \hookrightarrow \mathcal{O}} M_\tau,$$

and a small surjection $R' \to R$ whose kernel I is 1-dimensional over the field $R/\mathfrak{m}_R$, we have constructed a free R'-module M' together with a filtration $\{(M')^i\}$ and maps $\varphi^i_{M'}$ by lifting the M_τ. The filtration and $\{\varphi^i_{M'}\}$ make M' into a Fontaine-Laffaille module. There is an obvious $R' \otimes_{\mathbf{Z}_p} W$-module structure. The condition $M' = \sum_i \varphi^i_{M'}((M')^i)$ follows from Lemma 3.2.2.7. We also constructed a lift of the pairing $M' \times M' \to N$, and the filtration and $\varphi^i_{M'}$ are compatible with it (in the sense of Corollary 3.2.1.5) by our choice of $(\delta_{ih})_{ih}$. By Corollary 3.2.1.5 and Lemma 3.1.2.5, $T_{\mathrm{cris}}(M')$ gives a representation $\rho' : \Gamma_K \to G(R')$ lifting ρ.

3.2.3 Tangent Space

The final step in the proof of Theorem 3.2.1.2 is to analyze the tangent space of $D^{\mathrm{FL}}_{\overline{\rho}}$. It is a subspace $L^{\mathrm{FL}}_{\overline{\rho}}$ of the tangent space $H^1(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$ of deformations with fixed similitude character ν. We are mainly interested in its dimension as a vector space over k, and will analyze it by considering deformations ρ of $\overline{\rho}$ to the dual numbers $k[t]/(t^2)$. Recall that $G = \mathrm{GSp}_r$ (with even r) or $G = \mathrm{GO}_r$; let $\mathfrak{u}$ be the Lie algebra of the unipotent radical of a Borel subgroup of G.

Proposition 3.2.3.1. *Under the standing assumption that $\overline{\rho}$ is torsion-crystalline with pairwise distinct Fontaine-Laffaille weights for each $\tau : W \hookrightarrow \mathcal{O}$ contained in an interval of length $\frac{p-2}{2}$,*

$$\dim_k L_{\overline{\rho}}^{\mathrm{FL}} - \dim_k H^0(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) = [K : \mathbf{Q}_p] \dim_k \mathfrak{u}.$$

Let $\overline{V}$ be the Galois module given by $\overline{\rho}$, and for a lift ρ of $\overline{\rho}$ over $k[t]/(t^2)$ let V be the corresponding Galois module. The submodule tV is naturally isomorphic to $\overline{V}$, and we have an exact sequence

$$0 \to tV \to V \to \overline{V} \to 0.$$

Let $\overline{M}$ be the Fontaine-Laffaille module corresponding to $\overline{\rho}$, with pairing $\langle \cdot, \cdot \rangle : \overline{M} \times \overline{M} \to L_k$. We know $\overline{M}$ is a k-vector space of dimension $r[K : \mathbf{Q}_p]$. Let M be the Fontaine-Laffaille module corresponding to ρ. It is a free $k[t]/(t^2) \otimes W$-module, and fits in an exact sequence

$$0 \to tM \to M \to \overline{M} \to 0$$

of Fontaine-Laffaille modules. The map $\overline{M} \to tM$ induced by $t : M \to M$ is an isomorphism of Fontaine-Laffaille modules since it is so on underlying k-vector spaces by $k[t]/(t^2)$-freeness of M. As before, we have decompositions

$$M = \bigoplus_{\tau : W \hookrightarrow \mathcal{O}} M_\tau \quad \text{and} \quad \overline{M} = \bigoplus_{\tau : W \hookrightarrow \mathcal{O}} \overline{M}_\tau$$

from Lemma 3.1.2.3.

Using Lemma 3.2.2.5, pick a basis $\{v_{\tau,i}\}_{i=1}^r$ of the $k[t]/(t^2)$-module M_τ such that $v_{\tau,i}$ is a basis for a $k[t]/(t^2)$-complement to $M_\tau^{w_i+1}$ in $M_\tau^{w_i}$ and such that the pairing $M_{\sigma\tau} \times M_{\sigma\tau} \to L_{\sigma\tau}$ with respect to the $m_{\tau,i} := \varphi_M^{w_i}(v_{\tau,i})$ is ω_τ-times of the standard pairing. As 1-units admit square roots, we may assume that $\omega_\tau \in k^\times$. Note that $\{m_{\tau,i}\} \cup \{tm_{\tau,i}\}$ is a basis for $M_{\sigma\tau}$ as a k-vector space, and $\{m_{\tau,i}\}_{\tau,i}$ is a basis for M as a $k[t]/(t^2)$ module.

Let M_0 be the submodule of M spanned by the $\{v_{\tau,i}\}_{\tau,i}$ as a k-vector space. We have that $tM_0 = tM \simeq \overline{M}$ as vector spaces, and have an obvious decomposition

$$M_0 = \bigoplus_{\tau : W \hookrightarrow \mathcal{O}} M_{\tau,0}.$$

We obtain a filtration on M_0 by intersection: $M_{\tau,0}^i = M^i \cap M_{\tau,0}$. We also obtain a pairing $M_0 \times M_0 \to L$ by restriction.

Lemma 3.2.3.2. *We have that $M_\tau^i = M_{\tau,0}^i \otimes k[t]/(t^2)$, and hence $M^i = M_0^i \otimes k[t]/(t^2)$.*

Proof. We know that the $k[t]/(t^2)$-span of v_i is a $k[t]/(t^2)$-complement to $M_\tau^{w_i+1}$ in $M_\tau^{w_i}$. Hence $M_\tau^{w_i}/M_\tau^{w_i+1}$ is isomorphic to the k-span of v_i and tv_i. As the filtration is automatically split (i.e. M_τ^i is a direct summand of M_τ, and hence M_τ^i is a direct summand of M_τ^{i-1}), this suffices. ☐

Observe that the surjection of Fontaine-Laffaille modules $M \to \overline{M}$ carries M_0 isomorphically onto $\overline{M}$. Under the isomorphism of k-vector spaces $M_0 \to \overline{M}$, the pairing on M_0 and the pairing on $\overline{M}$ are identified because by choice of basis the pairing on M_0 is a $k^\times$-multiple of the standard pairing. Furthermore, extending the pairing $M_0 \times M_0 \to L$ by $k[t]/(t^2)$-bilinearity recovers the pairing on M. Using $M_0 \simeq \overline{M}$, we can also define $\varphi_{M_0}^i : M_0^i \to M_0$ to be the lift of $\varphi_{\overline{M}}^i$ to M_0^i. It is compatible with the pairing on M_0. Note that it is *not* the same as $\varphi_M^i|_{M_0^i}$.

Our goal is to describe the set of strict equivalence classes of deformations M of $\overline{M}$, so by making these identifications it remains to study ways to lift $\varphi_{\overline{M}}^i$ to a map $\varphi_{M_0 \otimes k[t]/(t^2)}^i : M_0^i \otimes k[t]/(t^2) \to M_0 \otimes k[t]/(t^2)$. For $n, n' \in M_0^i$ we may write

$$\varphi_M^i(n + tn') = \varphi_{M_0}^i(n) + t(\varphi_{M_0}^i(n') + \delta_i(n))$$

for some σ-semilinear $\delta_i : M_0^i \to M_0$ which completely determines φ_M^i. It is clear that for $n \in M_0^{i+1}$ we have $\delta_i(n) = 0$ due to the relation $\varphi_{M_0}^i(n) = p\varphi_{M_0}^{i+1}(n) = 0$. Thus, δ_i factors through M_0^i/M_0^{i+1}, and together the δ_i define a σ-semilinear

$$\delta : \mathrm{gr}^\bullet(M_0) \to M_0.$$

Compatibility with the pairing says exactly

$$\langle \varphi_M^i(n+tn'), \varphi_M^j(m+tm')\rangle = \varphi_L^{i+j}(\langle n+tn', m+tm'\rangle)$$

for $n, n' \in M_0^i$ and $m, m' \in M_0^j$ and all i and j. Expanding and using the compatibility of the $\varphi_{M_0}^i$ with the pairing, we see that it is necessary and sufficient that

$$\langle \delta_i(n), \varphi_{M_0}^j(m)\rangle + \langle \varphi_{M_0}^i(n), \delta_j(m)\rangle = 0 \tag{3.2.3.1}$$

for $n \in M_0^i$ and $m \in M_0^j$ and all i and j. As $\overline{M} = \sum_i \varphi_{\overline{M}}^i(\overline{M}^i)$ and we defined $\varphi_{M_0}^i$ to lift $\varphi_{\overline{M}}^i$, it follows that $M_0 = \sum_i \varphi_{M_0}^i(M_0^i)$. Furthermore, we have an isomorphism $\varphi : \mathrm{gr}^\bullet M_0 \to M_0$. This allows us to rewrite (3.2.3.1) as

$$\langle \delta'\varphi(n), \varphi(m)\rangle + \langle \varphi(n), \delta'\varphi(m)\rangle = 0$$

where δ' is the k-linear composition of φ^{-1} with δ. In other words,

$$\langle \delta' x, y\rangle + \langle x, \delta' y\rangle = 0$$

for all $x, y \in M_0$. Note that δ' is compatible with the filtration, the pairing, and the $k\otimes W$-module structure. Denote the collection of all such δ' by $\mathrm{End}_{k\otimes W}(M_0, \langle\cdot,\cdot\rangle)$: it is isomorphic to $\mathfrak{sp}_r(k\otimes W)$ or $\mathfrak{so}_r(k\otimes W)$, which have dimension $[K:\mathbf{Q}_p](\dim G - 1)$ over k.

Lemma 3.2.3.3. *For such a choice of δ', we obtain a Fontaine-Laffaille module $M \in \mathrm{MF}_{W,\mathrm{tor}}^f$ together with a pairing $M \times M \to L$ as in Corollary 3.2.1.5.*

Proof. This is just bookkeeping. First, observe that $\sum_i \varphi_M^i(M^i)$ is a $k[t]/(t^2)$-module containing $\varphi_{M_0}^i(M_0^i) = M_0$. Thus it is M. It is immediate that the pairing is compatible with the filtration. We chose δ' so that the pairing is compatible with the φ_M^i. □

Of course, different δ' may give isomorphic deformations of $\overline{M}$. Suppose that we are given δ and γ such that the Fontaine-Laffaille modules they create are strictly equivalent as deformations of $\overline{M}$. We have shown that the underlying module, pairing, and filtration can be identified with the fixed data $M = M_0 \otimes k[t]/(t^2)$, $\langle\cdot,\cdot\rangle \otimes k[t]/(t^2)$, and $M_0^i \otimes k[t]/(t^2)$ so that the isomorphism reduces to the identity modulo t (by strictness). This means there exists an isomorphism $\alpha : M_0 \to M_0$ compatible with the pairing, filtration, and module structure such that

$$(1+t\alpha)\left(\varphi_{M_0}^i(n) + t(\varphi_{M_0}^i(n') + \delta_i(n))\right) = \varphi_{M_0}^i(n) + t(\varphi_{M_0}^i(\alpha(n)+n') + \gamma_i(n)).$$

Simplifying, this is the condition that

$$\gamma_i(n) - \delta_i(n) = \alpha(\varphi_{M_0}^i(n)) - \varphi_{M_0}^i(\alpha(n)).$$

In other words, $\delta, \gamma \in \mathrm{End}(M_0, \langle\cdot,\cdot\rangle)$ define the same deformation if and only if $\gamma_i - \delta_i$ is of the form $\alpha \circ \varphi_{M_0}^i - \varphi_{M_0}^i \circ \alpha$ for all i and some $k\otimes W$-linear $\alpha : M_0 \to M_0$ that is compatible with the filtration, pairing, and module structures. Under the identification of $\mathrm{End}_{k\otimes W}(M_0, \langle\cdot,\cdot\rangle)$ with the Lie algebra of a symplectic or orthogonal group valued in $k\otimes W$, these are the elements in the Lie algebra of the Borel subgroup corresponding to the filtration. (The assumption that the Fontaine-Laffaille weights for each τ are pairwise distinct is what makes it a Borel subgroup.) This has dimension $[K:\mathbf{Q}_p](\dim B - 1)$ as a k-vector space, where B is a Borel subgroup of G.

Finally, we must understand when α and β satisfy

$$\alpha \circ \varphi_{M_0}^i - \varphi_{M_0}^i \circ \alpha = \beta \circ \varphi_{M_0}^i - \varphi_{M_0}^i \circ \beta.$$

This happens exactly when $\alpha - \beta$ commutes with the $\varphi_{M_0}^i$ (as well as being compatible with the filtration, pairing, and module structure). In other words, $\alpha - \beta \in \mathrm{End}_{\mathrm{MF}_W}(M_0, \langle\cdot,\cdot\rangle)$. But under T_{cris}, this is identified with endomorphisms of $\overline{\rho}$ preserving the pairing (not just up to a similitude factor), and in particular has dimension $\dim_k H^0(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$.

We can express this analysis as the exact sequence

$$0 \to \mathrm{End}_{\mathrm{MF}_W}(M_0, \langle \cdot, \cdot \rangle) \to \mathrm{Fil}^0 \left(\mathrm{End}_{k\otimes W}(M_0, \langle \cdot, \cdot \rangle)\right) \to \mathrm{End}_{k\otimes W}(M_0, \langle \cdot, \cdot \rangle) \to D^{\mathrm{FL}}_{\overline{\rho}}(k[t]/(t^2)) \to 0.$$

In particular, we see that the space of deformations of $\overline{\rho}$ to $k[t]/(t^2)$ has dimension

$$[K : \mathbf{Q}_p](\dim G - 1) - [K : \mathbf{Q}_p[(\dim B - 1) + \dim_k H^0(\Gamma_K, \mathrm{ad}^0(\overline{\rho})) = [K : \mathbf{Q}_p] \dim_k \mathfrak{u} + \dim_k H^0(\Gamma_K, \mathrm{ad}^0(\overline{\rho}))$$

(since the Borel subgroup of the derived group of G has codimension-1 in a Borel subgroup of G). This completes the proof of Proposition 3.2.3.1. □

Chapter 4

Minimally Ramified Deformations

Let $\ell \neq p$ be primes and L be a finite extension of $\mathbf{Q}_\ell$. For a split connected reductive group G over the valuation ring $\mathcal{O}$ of a p-adic field K with residue field k, consider a residual representation $\overline{\rho} : \Gamma_L \to G(k)$. The methods of Chapter 2 require a nice class of deformations of $\overline{\rho}$. If $\overline{\rho}$ were unramified, the unramified deformation condition (Example 2.2.2.9) would often work for our purposes. The interesting case is when $\overline{\rho}$ is ramified: we would like to define a deformation condition that "restricts the ramification" of lifts of $\overline{\rho}$ so the resulting deformation condition is liftable, despite the fact that the unrestricted deformation condition for $\overline{\rho}$ may not be liftable. To be precise, we require a deformation condition $\mathcal{D}_{\overline{\rho}}$ such that $\mathcal{D}_{\overline{\rho}}$ is liftable and has tangent space with dimension (at least) $\dim_k H^0(\Gamma_L, \mathrm{ad}^0(\overline{\rho}))$. We will obtain such a deformation condition after making an étale local extension of $\mathcal{O}$, which is harmless for our application.

In the case that $G = \mathrm{GL}_m$, the minimally ramified deformation condition defined in [CHT08, §2.4.4] works. We will generalize this to a *minimally ramified deformation condition* for symplectic and orthogonal similitude groups on m-dimensional spaces when $p \geq m$. The key inspiration comes from the arguments of [Tay08, §3], where a similar deformation condition for GL_m is analyzed by studying deformations of a nilpotent element of $\mathfrak{g}_k = \mathrm{Lie}\, G_k$. In §4.1, we review facts about nilpotent orbits over algebraically closed fields of very good characteristic and find "nice" integral representatives N for such orbits. In §4.2, we study the centralizer scheme $Z_G(N)$ over $\mathcal{O}$ and show that it is smooth. We can reduce checking $\mathcal{O}$-smoothness to the problem of finding elements $g \in Z_G(N)(\mathcal{O})$ such that g_k lies in any specified component of $Z_{G_k}(N_k)/Z_{G_k}(N_k)^\circ$. Then in §4.3 we define the notion of a "pure nilpotent" deforming N_k and study the space of such deformations. Finally in §4.4 and §4.5 we define the minimally ramified deformation condition using our study of nilpotents, first for a special class of tamely ramified representations and then in general.

The "minimally ramified" deformation condition is defined only for symplectic and orthogonal similitude groups, essentially because of an argument used in §4.5 to reduce the study of general representations to a special class of tamely ramified representations. We have adapted the argument of [CHT08, §2.4.4] to keep track of a symmetric or alternating pairing, so this is genuinely specific to symplectic or orthogonal groups. The arguments in earlier sections are phrased in such a way as to easily generalize to any split connected reductive $\mathcal{O}$-group G, given certain general facts about nilpotent orbits and centralizers of nilpotent elements. In particular, we use the following:

(N1) for every nilpotent orbit of $G(\overline{k})$ in $\mathfrak{g}_{\overline{k}}$, there exists $N \in \mathfrak{g}$ such that both $N_{\overline{K}} \in \mathfrak{g}_{\overline{K}}$ and $N_{\overline{k}} \in \mathfrak{g}_{\overline{k}}$ "lie in that nilpotent orbit" (using the combinatorial parametrization of nilpotent orbits discussed in §4.1 to relate nilpotent orbits in $\mathfrak{g}_{\overline{k}}$ and $\mathfrak{g}_{\overline{K}}$) and such that $Z_{G_k}(N_k)$ and $Z_{G_K}(N_K)$ are smooth;

(N2) for a suitable such $N \in \mathfrak{g}$ and every component of $Z_{G_{\overline{k}}}(N_{\overline{k}})$ there exists $g \in Z_G(N)(\mathcal{O})$ such that g_k lies in that component (which will imply the flatness and hence smoothness of the scheme-theoretic centralizer $Z_G(N)$ over $\mathcal{O}$);

(N3) for tame $\overline{\rho}$ and a tame inertial generator τ, there is a way to pass between a nice class of deformations of $N_k = \overline{\rho}(\tau)$ and deformations of $\overline{\rho}$ as made precise in §4.4. In particular, this uses the exponential map (4.4.1.1) to convert between nilpotents and unipotents, as well as elements $\Phi \in G(\mathcal{O})$ such that $\mathrm{ad}_G(\Phi)N = qN$ for some specified $q \in \mathbf{Z}$.

We will check all of these assumptions for GL_m, GSp_m, and GO_m when $p \geq m$, which suffices for our applications. It would be of interest to give uniform proofs, or at least to verify them for the exceptional groups (where there are just finitely many nilpotent orbits to check).

4.1 Representatives for Nilpotent Orbits

4.1.1 Algebraically Closed Fields

In this subsection, we let k be algebraically closed of characteristic $p \geq 0$ and take G to be a connected reductive group over k with p good for G. (By convention, characteristic zero is good for all G.) Let $\mathfrak{g} = \mathrm{Lie}\, G$. For a nilpotent $N \in \mathfrak{g}$, the orbit O_N of N under G can be defined as the locally closed (reduced) image of the orbit map through N. It is a smooth locally closed subscheme of $\mathfrak{g}$ by the closed orbit lemma, and is called a *nilpotent orbit* (for G). As G is connected, this orbit is irreducible. Let $Z_G(N)$ be the scheme-theoretic centralizer of a nilpotent N, representing the functor

$$R \mapsto \{g \in G(R) : \mathrm{Ad}_G(g)N_R = N_R\}$$

for k-algebras R.

In good characteristic, the finite number of nilpotent orbits can be described by the Bala-Carter method as we review below. (More information can be found in [Jan04, §4], and a uniform proof without case-checking in small characteristic is due to Premet [Pre03].) To state this, we need to define some terminology. Let H be a connected reductive k-group with p good for H, and $\mathfrak{h} = \mathrm{Lie}\, H$.

- A nilpotent $N \in \mathfrak{h}$ is a *distinguished nilpotent* if each torus contained in $Z_H(N)$ is contained in the center of H.
- For a parabolic $P \subset H$ with unipotent radical U, the *Richardson orbit* associated to P is the unique nilpotent orbit of H with dense intersection with $\mathrm{Lie}\, U$. Its intersection with $\mathrm{Lie}\, P$ is a single orbit under P.
- A parabolic subgroup $P \subset H$ with unipotent radical U is a *distinguished parabolic* if $\dim P/U = \dim U/\mathscr{D}(U)$.

Bala and Carter classified nilpotent orbits when the characteristic is good. One can check that if p is good for G, it will also be good for any Levi factor of a parabolic subgroup of G.

Fact 4.1.1.1. *If p is a good prime for G, the nilpotent orbits for G are in bijection with $G(k)$-conjugacy classes of pairs (L, P) where L is a Levi factor of a parabolic subgroup of G and P is a distinguished parabolic of L. The nilpotent orbit for G associated to (L, P) is the unique one meeting* $\mathrm{Lie}(P)$ *in its Richardson orbit for L.*

Example 4.1.1.2. For $G = \mathrm{GL}_n$, conjugacy classes of parabolic subgroups of GL_n are indexed by partitions $n = n_1 + \ldots + n_r$, with a Levi subgroup given by the product $\mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_r}$ for the associated "standard" parabolic subgroup. The only distinguished parabolic subgroups of GL_{n_i} are the Borel subgroups. Taking into account conjugation, we conclude that conjugacy classes of nilpotents are in bijection with partitions of n. A representative for each orbit is given by a block matrix in Jordan canonical form with eigenvalues zero and blocks of size $n_1, \ldots, n_r$. This is worked out in detail in [Jan04, §4.1,4.4,4.8]. The nilpotent orbits are the same for SL_n [Jan04, §1.2].

The *Bala-Carter data* $\mathcal{C}$ for G is the set of $G(k)$-conjugacy classes of pairs (L, P) as above. It turns out $\mathcal{C}$ is independent of k in the sense that it can be described completely in terms of the root datum of G as follows. All Levi subgroups L of a parabolic k-subgroup Q of G are a single $\mathcal{R}_{u,k}(Q)$-orbit, so in Fact 4.1.1.1 we may restrict to one Q per $G(k)$-conjugacy class and one L per Q. We may pick L so that it contains a (split) maximal torus T. After conjugation by $L(k)$, the distinguished parabolic subgroup $P \subset L$ may be assumed to contain T as well. But we know that parabolic subgroups Q of G containing T are in bijection with parabolic subsets of $\Phi(G, T)$ via $Q \mapsto \Phi(Q, T)$, so the possible Levi factors L of Q containing T are described just in terms of the root datum. Likewise, parabolic subgroups P of L containing T are in bijection with parabolic subsets of $\Phi(L,T)$. If we can characterize the condition that P is distinguished just in terms of the root data, this would mean that the Bala-Carter data can be described solely in terms of the root data and so is completely combinatorial.

We do so by constructing a grading on the Lie algebra of a parabolic P. Pick a Borel subgroup $B \subset G$ satisfying $T \subset B \subset P$. Let $\mathfrak{t} = \mathrm{Lie}(T)$ and $\Delta \subset \Phi = \Phi(L,T)$ be the set of positive simple roots determined by B. There is a unique subset $I \subset \Delta$ such that $P = BW_IB$ where W_I is the subset of the Weyl group generated by reflections with respect to roots in I. Define a group homomorphism $f : \mathbf{Z}\Phi \subset \mathbf{Z}^\Delta \to \mathbf{Z}$ by specifying that on the basis Δ we have

$$f(\alpha) = \begin{cases} 2 & \alpha \in \Delta - I, \\ 0 & \alpha \in I. \end{cases}$$

This function gives a grading on $\mathfrak{l} = \mathrm{Lie}(L)$:

$$\mathfrak{l}(i) = \bigoplus_{f(\alpha)=i} \mathfrak{l}_\alpha \quad \text{and} \quad \mathfrak{l}(0) = \left(\bigoplus_{f(\alpha)=0} \mathfrak{l}_\alpha \right) \oplus \mathfrak{t}$$

(sums indexed by $\alpha \in \Phi$). With respect to this grading,

$$\mathrm{Lie}\, P = \bigoplus_{i \geq 0} \mathfrak{l}(i) \quad \text{and} \quad \mathrm{Lie}\, U = \bigoplus_{i > 0} \mathfrak{l}(i).$$

The condition that P is distinguished is equivalent to the condition that

$$\dim \mathfrak{l}(0) = \dim \mathfrak{l}(2) + \dim Z_L$$

by [Car85, Corollary 5.8.3] as p is good for L. But this condition depends only on the root datum. Thus the Bala-Carter data for G can be described in a manner independent of the choice of algebraically closed field.

Definition 4.1.1.3. For $\sigma \in \mathcal{C}$, let $O_\sigma \subset \mathfrak{g}$ (or $O_{k,\sigma}$ if it is necessary to specify the field) be the corresponding nilpotent orbit.

From the classification of nilpotent orbits over algebraically closed fields, it is known that the corresponding nilpotent orbits in characteristic zero and characteristic p have the same dimension.

Example 4.1.1.4. Let $G = \mathrm{GL}_n$ over an algebraically closed field k. For a partition $n = n_1 + n_2 + \ldots + n_r$, define $d_i(\sigma)$ inductively by $d_0(\sigma) = 0$ and $d_i(\sigma) = d_{i-1}(\sigma) + \#\{j : n_j \geq i\}$. The orbit closure $\overline{O}_\sigma$ consists of nilpotents $N \in \mathfrak{g}$ such that the $(n+1-d_i(\sigma)) \times (n+1-d_i(\sigma))$ minors of N^i vanish for all $i = 1, \ldots, n$.

Remark 4.1.1.5. We consider connected reductive groups, but natural groups like O_n are disconnected. The nilpotent orbits of O_n and SO_n are different but closely related; as explained in [Jan04, §1.12], the only change is that certain pairs of orbits for SO_n may become a single O_n-orbit.

4.1.2 Representatives of Nilpotent Orbits

Now return the case when G is a split reductive group scheme with connected fibers over a complete discrete valuation ring $\mathcal{O}$ with $\mathfrak{g} = \mathrm{Lie}\, G$. Let k be the residue field of characteristic $p > 0$. Assume p is very good for G_k. For $\sigma \in \mathcal{C}$, we seek elements

$$N_\sigma \in \mathfrak{g} \text{ such that } (N_\sigma)_k \in O_{\overline{k},\sigma} \text{ and } (N_\sigma)_K \in O_{\overline{K},\sigma}. \qquad (4.1.2.1)$$

In the case $G = \mathrm{GL}_n$, we will see that the standard representatives in $G(\mathcal{O})$ for nilpotent orbits in Jordan canonical form satisfy this condition. We will also explicitly describe such N_σ for symplectic and orthogonal groups, as we will need this concrete description to analyze the centralizer $Z_G(N)$ as an $\mathcal{O}$-scheme.

Remark 4.1.2.1. In general, one can build such an N_σ using the root system as follows. Let T be a split maximal torus of G. Pick a pair (L, P) consisting of a Levi factor L of a parabolic $\mathcal{O}$-subgroup of G and a fiber-wise distinguished parabolic $\mathcal{O}$-subgroup P of L corresponding to σ such that $T \subset P$. For any subset $J \subset \Phi(L, T)$, consider the element

$$N_J := \sum_{\alpha \in J} N_\alpha \in \mathfrak{g}$$

where N_α is an $\mathcal{O}$-basis for the root line $\mathfrak{g}_\alpha \subset \mathfrak{g}$. For any $\sigma = (L, P)$, according to [SS70, III.4.29] there is some subset J of a positive system of roots in $\Phi(G, T)$ with $|J| < \dim(T)$ for which N_J lies in $O_{\overline{k},\sigma}$ and $O_{\overline{K},\sigma}$. (The method used to prove [SS70, III.4.29] requires a stronger condition on p, as the Jacobson-Morozov theorem is used in the analysis of nilpotent orbits presented in [SS70]. There should be no difficulty in removing this assumption, but we don't treat this as our final results only apply to symplectic and orthogonal groups, for which we make direct constructions below.)

For GL_n, it is easy to describe an N_σ as in (4.1.2.1) in terms of the root data via Jordan canonical form:

Example 4.1.2.2. Let $G = \mathrm{GL}_n$ with nilpotent orbit corresponding to the partition $n_1 + n_2 + \ldots + n_r = n$. For the diagonal torus T and standard upper triangular Borel subgroup B, let e_i be the character extracting the ith diagonal entry of the torus. The positive simple roots in $\Phi(G, T)$ with respect to B are $\{\alpha_i = e_i - e_{i+1} : i = 1, \ldots, n-1\}$. For a partition $n = n_1 + \ldots + n_r$, the corresponding nilpotent orbit as in Example 4.1.1.2 corresponds to the standard upper triangular Borel subgroup B_L of $L = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \ldots \times \mathrm{GL}_{n_r}$. Now $\Phi(L, T)$ contains as its set Δ_L of simple roots with respect to B_L those $e_i - e_{i+1}$ with $i \neq n_1, n_1 + n_2, \ldots n_r$ (in other words, i and $i+1$ lie in the "same block"). We consider

$$N = \sum_{\alpha \in \Delta_L} N_\alpha$$

where N_α is a basis element for the root line $\mathfrak{g}_\alpha$. Identifying $\mathfrak{g}_k$ with $\mathrm{End}(k^n)$, N_k is the nilpotent matrix in Jordan canonical form whose blocks (in order) are of sizes $n_1, n_2, \ldots, n_r$. Thus, N satisfies (4.1.2.1).

For symplectic and orthogonal groups, explicitly describing the nilpotents constructed from Bala-Carter data is more complicated. We will sketch the approach, but not carry out the details because it turns out to be more convenient to construct and analyze representatives in these cases using a partition-based classification rather than via Bala-Carter data; this approach gives additional information about the centralizers that will be needed in §4.2.2.

Example 4.1.2.3. Consider the groups Sp_m with $m = 2n$, or SO_m with $m = 2n$ or $m = 2n+1$ (so $[\frac{m}{2}] = n$) and assume $p \neq 2$ and $n \geq 2$. We will first describe the Levi subgroups of parabolic subgroups [Jan04, §4.5]. Choose an integer $0 < n_0 \leq m$ with $n_0 \equiv m \pmod 2$. In the orthogonal case, assume that $n_0 \neq 2$. Choose a partition $2d_1 + 2d_2 + \ldots + 2d_r \geq 0$ of $m - n_0$ into even parts. Choose a partition

$$\{1, \ldots, n\} = \overline{S}_0 \coprod \overline{S}_1 \coprod \ldots \coprod \overline{S}_r$$

where $|\overline{S}_i| = d_i$ for $1 \leq i \leq r$ and hence $|\overline{S}_0| = [n_0/2]$. For $1 \leq i \leq r$, let $S_i = \overline{S}_i \cup (n + \overline{S}_i)$. If m is even, let $S_0 = \overline{S}_0 \cup (n + \overline{S}_i)$, while if m is odd let $S_0 = \overline{S}_0 \cup (n + \overline{S}_i) \cup \{0\}$. Clearly $\{S_i\}$ is a partition of $\{1, 2, \ldots, 2n\}$ if $m = 2n$ and of $\{0, 1, 2, \ldots, 2n\}$ if $m = 2n+1$.

Let $\{v_i\}$ be the standard basis k^m. Define $W_i = \mathrm{span}_{j \in \overline{S}_i} v_j$, $\overline{W}_i = \mathrm{span}_{j \in \overline{S}_i} v_{n+j}$, and $V_i = \mathrm{span}_{j \in S_i} v_j$. This gives an orthogonal direct sum decomposition

$$V_0 \oplus V_1 \oplus V_2 \ldots \oplus V_r = k^m$$

with respect to the standard pairings. For $1 \leq i \leq r$, we have $V_i = W_i \oplus \overline{W}_i$ with subspaces W_i and $\overline{W}_i$ in perfect duality via the chosen standard pairing, and an automorphism of W_i covariantly induces an automorphism of $\overline{W}_i$ via "dual-inverse", giving an inclusion of $\mathrm{GL}(W_i)$ inside $\mathrm{SO}(V_i)$ or $\mathrm{Sp}(V_i)$. The subgroup

$$L = \mathrm{Sp}(V_0) \times \mathrm{GL}(W_1) \times \ldots \times \mathrm{GL}(W_r)$$

is a Levi subgroup of a parabolic subgroup in Sp_m. The subgroup

$$L = \mathrm{SO}(V_0) \times \mathrm{GL}(W_1) \times \ldots \times \mathrm{GL}(W_r)$$

is a Levi subgroup of a parabolic subgroup in SO_m. Up to conjugation, all Levi subgroups of parabolic subgroups arise in this way [Jan04, §4.5].

A distinguished parabolic subgroup of L is a direct product of distinguished parabolic subgroups of $\mathrm{Sp}(V_0)$ or $\mathrm{SO}(V_0)$ and each of the $\mathrm{GL}(W_i)$. There is a unique conjugacy class of distinguished parabolic subgroups in $\mathrm{GL}(W_i)$, namely the Borel subgroups. The conjugacy classes of distinguished parabolic subgroups in $G_0 = \mathrm{Sp}(V_0)$ or $\mathrm{SO}(V_0)$ are described in the tables of [Car85, §5.9]. There are several classes, and it will be much more convenient to describe them using an approach based on partitions.

Now we provide partition-based classifications for symplectic and orthogonal groups, similar to the classification for GL_m we have already seen in Example 4.1.1.2. We will use them to construct elements as in (4.1.2.1). Let $G = \mathrm{Sp}_m$ with $m = 2n$, or $G = \mathrm{O}_m$ with $m = 2n$ or $m = 2n + 1$. As usual, we assume $n \geq 2$. Recall that Sp_m and O_m are defined using standard pairings on a free $\mathcal{O}$-module M of rank m. For $m = 2n$, the *standard alternating pairing* φ_{std} on $\mathcal{O}^m$ is the one given by the block matrix

$$\begin{pmatrix} 0 & I'_n \\ -I'_n & 0 \end{pmatrix},$$

where I'_n denotes the anti-diagonal matrix with 1's on the diagonal. The *standard symmetric pairing* φ_{std} on $\mathcal{O}^m$ is the one given by the matrix I'_m.

Remark 4.1.2.4. We chose to work with O_m instead of SO_m, as the classification is cleaner for O_m. The nilpotent orbits are almost the same for SO_m, except that certain nilpotent orbits of O_m (the ones where the partition contains only even parts) split into two SO_m-orbits [Jan04, Proposition 1.12] (conjugation by an element of O_m with determinant -1 carries one such orbit into the other).

Definition 4.1.2.5. Let σ denote a partition $m = m_1 + m_2 + \ldots + m_r$ of m. It is *admissible* if

- every even m_i appears an even number of times when $G = \mathrm{O}_m$;
- every odd m_i appears an even number of times when $G = \mathrm{Sp}_m$.

The admissible partitions of m are in bijection with nilpotent orbits of Sp_m or O_m over an algebraically closed field [Jan04, Theorem 1.6]. The corresponding orbit is the intersection of $\mathfrak{g} \subset \mathfrak{gl}_m$ with the GL_m-orbit corresponding to that partition of m. Note that GL_m-orbit representatives in Jordan canonical form need *not* lie in $\mathfrak{g}$.

We will construct nilpotents together with a pairing, and then show how to relate the constructed pairing to the standard pairings used to define G. Let $\epsilon = 1$ in the case of O_m, and $\epsilon = -1$ in the case of Sp_m.

Definition 4.1.2.6. Let $d \geq 2$ be an integer. Define $M(d) = \mathcal{O}^d$, with basis $v_1, \ldots v_d$ and a perfect symmetric or alternating pairing φ_d such that

$$\varphi_d(v_i, v_j) = \begin{cases} (-1)^i, & i + j = d + 1 \\ 0, & \text{otherwise} \end{cases}$$

(alternating for even d, symmetric for odd d). Define a nilpotent $X_d \in \mathrm{End}(M(d))$ by $X_d v_i = v_{i-1}$ for $1 < i \leq d$ and $X_d v_1 = 0$.

Similarly, define $M(d, d) = \mathcal{O}^{2d}$ with basis $v_1, \ldots v_d, v'_1, \ldots, v'_d$ and a perfect symmetric or alternating pairing $\varphi_{d,d}$ by extending

$$\varphi_{d,d}(v_i, v_j) = \varphi_{d,d}(v'_i, v'_j) = 0 \quad \text{and} \quad \varphi_{d,d}(v_i, v'_j) = \begin{cases} (-1)^i, & i + j = d + 1 \\ 0, & \text{otherwise} \end{cases}$$

Define a nilpotent $X_{d,d} \in \mathrm{End}(M(d, d))$ by $X_{d,d} v_i = v_{i-1}$ and $X_{d,d} v'_i = v'_{i-1}$ for $1 < i \leq d$, and $X_{d,d} v_1 = X_{d,d} v'_1 = 0$.

It is straightforward to verify the pairings are perfect and that X_d and $X_{d,d}$ are skew with respect to the corresponding pairing. The pairing $\varphi_{d,d}$ can be symmetric or alternating. Given an admissible partition $\sigma : m = m_1 + m_2 + \ldots + m_r$, we will construct a free $\mathcal{O}$-module of rank m with a symmetric or alternating pairing respectively and a nilpotent endomorphism respecting that pairing such that the Jordan block structure of nilpotent endomorphism on the geometric special fiber over $\operatorname{Spec}\mathcal{O}$ is given by σ. Let $n_i(\sigma) = \#\{j : m_j = i\}$.

- If $G = \mathrm{O}_m$ then $n_i(\sigma)$ is even for even i, so we can define
$$M_\sigma = \bigoplus_{i\,\mathrm{odd}} M(i)^{\oplus n_i(\sigma)} \oplus \bigoplus_{i\,\mathrm{even}} M(i,i)^{\oplus n_i(\sigma)/2}.$$
- If $G = \mathrm{Sp}_m$ then $n_i(\sigma)$ is even for odd i, so we can define
$$M_\sigma = \bigoplus_{i\,\mathrm{odd}} M(i,i)^{\oplus n_i(\sigma)/2} \oplus \bigoplus_{i\,\mathrm{even}} M(i)^{\oplus n_i(\sigma)}.$$

Let φ_σ and X_σ denote the pairing and nilpotent endomorphism defined by the pairing and nilpotent endomorphism on each piece using Definition 4.1.2.6. In all cases, M_σ is a free $\mathcal{O}$-module of rank m. For each σ, let G_σ be the automorphism scheme $\underline{\mathrm{Aut}}(M_\sigma, \varphi_\sigma)$, so for an algebraically closed field F over $\mathcal{O}$ we have an isomorphism $(G_\sigma)_F \simeq G_F$ well-defined up to $G(F)$-conjugation by using F-linear isomorphisms $(M_\sigma, \varphi_\sigma) \simeq (F^m, \varphi_{\mathrm{std}})$. Thus, the element $X_\sigma \in \operatorname{Lie} G_\sigma$ goes over to a $G(F)$-orbit in $\mathfrak{g}_F$.

Let $e_1, e_2, \ldots e_m$ be the standard basis for $\mathcal{O}^m$. The elements e_i and e_{m+1-i} pair non-trivially under the standard pairing. When $m = 2n+1$, e_{n+1} pairs non-trivially with itself under the standard pairing. We now relate the standard pairings to the pairings φ_σ.

Lemma 4.1.2.7. *For all admissible partitions of m, the specializations of the X_σ at geometric points ξ of $\operatorname{Spec}\mathcal{O}$ constitute a set of representatives for the nilpotent orbits of G_ξ.*

Proof. The set of admissible partitions of m is in bijection with the set of nilpotent orbits over any algebraically closed field [Jan04, Theorem 1.6]. The X_σ we constructed are integral versions of the representatives constructed in [Jan04, §1.7]. □

Proposition 4.1.2.8. *Suppose that $\sqrt{-1}, \sqrt{2} \in \mathcal{O}^\times$. Then φ_σ is equivalent to the standard pairing over $\mathcal{O}$. There exists an $\mathcal{O}$-basis $\{v_i\}$ of M_σ with respect to which the pairing is given by φ_σ and X_σ satisfies the condition in* (4.1.2.1).

Proof. The standard pairings are very similar to φ_σ. In the case of Sp_m, each basis vector pairs trivially against all but one other basis vector, with which it pairs as ± 1. So after reordering the basis, φ_σ is the standard pairing. The case of O_m is slightly more complicated. Let $\sigma : m = m_1 + m_2 + \ldots + m_r$ be an admissible partition. The construction of M_σ and φ_σ gives a basis $\{v_{i,j}\}$ where $1 \leq i \leq r$ and $1 \leq j \leq m_i$. From the construction of φ_σ, we see that $v_{i,j}$ pairs trivially against all basis vectors except for v_{i,m_i+1-j}. So as long as $2j \neq m_i + 1$, we obtain a pair of basis vectors which are orthogonal to all others and which pair to ± 1. For each odd m_i, the vector $v_{i,(m_i+1)/2}$ pairs non-trivially with itself. The standard pairing with respect to the basis e_i has such a vector only when $m = 2n+1$ and then only for one e_i.

We must change the basis over $\mathcal{O}$ so that φ_σ becomes the standard symmetric pairing. Let $v = v_{i,(m_i+1)/2}$ and $v' = v_{j,(m_j+1)/2}$ be two distinct vectors which pair non-trivially with themselves. In particular, $\varphi_\sigma(v,v) = (-1)^{(m_i+1)/2} := \eta$ and $\varphi_\sigma(v',v') = (-1)^{(m_j+1)/2} := \eta'$. Define
$$w = \frac{\sqrt{\eta}v - \sqrt{-\eta'}v'}{\sqrt{2}} \quad \text{and} \quad w' = \frac{\sqrt{\eta}v + \sqrt{-\eta'}v'}{\sqrt{2}}.$$
Then we see that $\varphi_\sigma(w,w) = 0 = \varphi_\sigma(w',w')$ and $\varphi_\sigma(w,w') = 1$. Making this change of variable over $\mathcal{O}$ (which requires $\sqrt{-1}, \sqrt{2} \in \mathcal{O}^\times$), we have reduced the number of basis vectors which pair non-trivially with themselves by two, and produced a new pair of basis vectors orthogonal to the others and which pair to 1. By induction, we may therefore pick a basis $v'_1, \ldots, v'_m$ for which at most one basis vector pairs non-trivially with itself under φ_σ. After re-ordering, we may further assume that $\varphi_\sigma(v'_i, v'_j) = 0$ unless $i + j = m+1$, in which case $\varphi_\sigma(v'_i, v'_j) = \pm 1$. Suppose $j = m+1-i$. If $i \neq j$, by scaling v'_i we may assume that $\varphi_\sigma(v'_i, v'_j) = 1$. If $i = j$, we already know that $\varphi_\sigma(v'_i, v'_j) = 1$. With respect to this basis, φ_σ is the standard pairing. □

4.2 Smoothness of Centralizers of Pure Nilpotents

Keeping the notation of §4.1.2, we next study the centralizer $Z_G(N_\sigma)$ in more detail where $N_\sigma \in \mathfrak{g}$ is an element satisfying (4.1.2.1). In particular, this centralizer will be shown to be smooth when G is symplectic or orthogonal. We first review the known theory over fields, and then develop and apply a technique to deduce smoothness over $\mathcal{O}$ (i.e. $\mathcal{O}$-flatness) from the known smoothness in the field case.

4.2.1 Centralizers over Fields

In this section, let k be an algebraically closed field, G be a connected reductive group over k, and N a nilpotent element of $\mathfrak{g} = \operatorname{Lie} G$. As the formation of the scheme-theoretic centralizer commutes with base change, smoothness results for $Z_G(N)$ over k will imply such results over general fields (not necessarily algebraically closed).

The group scheme $Z_G(N)$ is the fiber over $0 \in \mathfrak{g}$ of the composition

$$G \xrightarrow{\operatorname{Ad}_G} \operatorname{GL}(\mathfrak{g}) \xrightarrow{T \mapsto TN - N} \mathfrak{g}.$$

Hence $\operatorname{Lie} Z_G(N)$ is the kernel of

$$\mathfrak{g} \xrightarrow{\operatorname{ad}_\mathfrak{g}} \operatorname{End}(\mathfrak{g}) \xrightarrow{T \mapsto TN} \mathfrak{g}$$

which is the Lie algebra centralizer $\mathfrak{z}_\mathfrak{g}(N)$. Recall that O_N denotes the smooth locally closed orbit $G.N \subset \mathfrak{g}$.

Lemma 4.2.1.1. *The following are equivalent:*

1. *The centralizer $Z_G(N)$ is smooth.*
2. *The map $\operatorname{ad}(N) : T_e(G) \to T_N(O_N)$ is surjective.*
3. *The orbit map $\mu_N : G \to O_N$ is smooth.*

Proof. The tangent space of $Z_G(N)$ at e is the kernel of $d\mu_N : T_e(G) \to T_N(O_N)$, so $Z_G(N)$ is smooth if and only if this kernel has the same dimension as $Z_G(N)$. As G and O_N are smooth and we have $\dim G - \dim Z_G(N) = \dim O_N$, dimension considerations force $d\mu_N$ to be surjective when $Z_G(N)$ is smooth. Thus (1) implies (2). To check that a morphism between smooth schemes is smooth, it suffices to check the map is surjective on tangent spaces. We need only check at the identity because of translations, so (2) implies (3). As $Z_G(N)$ is the fiber over $N \in O_N$, (3) implies (1). □

Remark 4.2.1.2. In references using the language of varieties rather than schemes (such as [Jan04]), $Z_G(N)$ is usually *defined* to be reduced (via its geometric points) and hence smooth, so the content of the analogue of Lemma 4.2.1.1 in such references is that the variety $Z_G(N)$ has Lie algebra $\mathfrak{z}_\mathfrak{g}(N)$; saying exactly that this variety agrees with the scheme-theoretic centralizer.

In a wide range of situations, all nilpotent centralizers are smooth. A direct calculation shows that this holds for $G = \operatorname{GL}_n$ (see [Jan04, §2.3]), and a criterion of Richardson leverages this to many other cases:

Proposition 4.2.1.3. *Suppose G is a smooth closed subgroup of $\operatorname{GL}(V)$ and there exist a subspace $W \subset \mathfrak{gl}(V)$ such that $\mathfrak{gl}(V) = \mathfrak{g} \oplus W$ and $[\mathfrak{g}, W] \subset W$. Then $Z_G(N)$ is smooth for any nilpotent $N \in \mathfrak{g}$.*

Proof. Let $\widetilde{\mathcal{O}}_N$ be the orbit of N under $\operatorname{GL}(V)$ in $\mathfrak{gl}(V)$, and O_N the orbit of N under G. We know that $T_N(\widetilde{\mathcal{O}}_N) = [\mathfrak{gl}(V), N]$ by Lemma 4.2.1.1(2) and the known smoothness of the centralizer for $\operatorname{GL}(V)$. The decomposition $\mathfrak{gl}(V) = \mathfrak{g} \oplus W$ then implies that

$$T_N(\widetilde{\mathcal{O}}_N) = [\mathfrak{g}, N] + [W, N].$$

But $[W, N] \subset W$, so $T_N(\widetilde{\mathcal{O}}_N) \cap \mathfrak{g} \subset [\mathfrak{g}, N] \subset T_N(O_N)$. On the other hand, $T_N(O_N)$ is obviously inside $T_N(\widetilde{\mathcal{O}}_N)$ and $\mathfrak{g}$. This implies that

$$[\mathfrak{g}, N] = T_N(O_N).$$

Then Lemma 4.2.1.1(2) gives smoothness. □

The criterion in Proposition 4.2.1.3 automatically holds in characteristic 0 if G is reductive, because $\mathfrak{g}$ has a complement as a $\mathfrak{g}$-representation in $\mathfrak{gl}(V)$; of course, this is not interesting because the smoothness conclusion holds for all group schemes of finite type over a field of characteristic zero. In characteristic $p > 0$, the criterion in Proposition 4.2.1.3 can checked in many cases [Jan04, §2.6], such as:

- for SL_n when $p \nmid n$ using the embedding $\mathrm{SL}(V) \hookrightarrow \mathrm{GL}(V)$ and taking $W = k \cdot \mathrm{id}_V$;
- for a simple adjoint exceptional groups G when p is good for G;
- for $\mathrm{SO}(V, \varphi)$ or $\mathrm{Sp}(V, \varphi)$ when $p \neq 2$ using the natural embeddings into $\mathrm{GL}(V)$ and taking $W = \{f \in \mathfrak{gl}(V) : f^* = f\}$.

(Here the non-degenerate pairing φ defines an adjoint f^* for any $f \in \mathfrak{gl}(V) = \mathrm{End}(V)$ via the requirement $\varphi(fv, w) = \varphi(v, f^*w)$.)

The same techniques work to prove the same result GSp_{2n} and GO_n; there does not appear to be a reference in the literature, so we now give a proof. Let V be a vector space over k with non-degenerate bilinear form $\varphi : V \times V \to k$ that is symmetric or alternating. Suppose $p \neq 2$.

Proposition 4.2.1.4. *If $p \nmid \dim(V)$, the group $G = \mathrm{GO}(V, \varphi)$ (respectively $G = \mathrm{GSp}(V, \varphi)$) with the natural inclusion into $\mathrm{GL}(V)$ admits a decomposition*

$$\mathfrak{gl}(V) = \mathfrak{g} \oplus M$$

with $[\mathfrak{g}, M] \subset M$, so the G-centralizer of any nilpotent $N \in \mathfrak{g}$ is smooth.

Proof. Let's recall why $\mathrm{tr}(f^*) = \mathrm{tr}(f)$ for any $f \in \mathfrak{gl}(V) = \mathrm{End}(V)$. Fix a basis of V Representing the pairing φ as an (invertible) matrix J and letting A and A^* be the matrices for f and f^*, the identity $\varphi(fv, w) = \varphi(v, f^*w)$ says

$$v^T A^T J w = v^T J A^* w$$

for all $v, w \in V$. This implies that $A^* = J^{-1} A^T J$, which has the same trace as A.

We can express the Lie algebra as $\mathfrak{g} = \{f \in \mathfrak{gl}(V) : f + f^* \in k \cdot \mathrm{Id}\}$. As $p \nmid \dim(V)$, we can define a complement to the scalar matrices by requiring the trace to be 0: define

$$M = \{f \in \mathfrak{gl}(V) : \mathrm{tr}(f) = 0 \quad \text{and} \quad f = f^*\}.$$

For $f \in \mathfrak{g}$, define $\lambda_f \in k$ by $f + f^* = \lambda_f \,\mathrm{Id}$. We compute that

$$\mathfrak{g} \cap M = \{f \in \mathfrak{gl}(V) : 2f = \lambda_f \,\mathrm{Id} \quad \text{and} \quad \mathrm{tr}(f) = 0\} = \{0\}.$$

On the other hand, $\mathfrak{g} + M = V$: the scalar matrices are in $\mathfrak{g}$ (G is a similitude group) and if $\mathrm{tr}(f) = 0$ then $\frac{f - f^*}{2} \in \mathfrak{g}$ and $\frac{f + f^*}{2} \in M$. Thus we have a decomposition

$$\mathfrak{gl}(V) = \mathfrak{g} \oplus M.$$

For $f \in \mathfrak{g}$ and $g \in M$, we compute that $\mathrm{tr}([f, g]) = 0$ and

$$(fg - gf)^* = g^* f^* - f^* g^* = g(-f + \lambda_f \,\mathrm{Id}) - (-f + \lambda_f \,\mathrm{Id})g = fg - gf.$$

Therefore $[\mathfrak{g}, M] \subset M$ as desired. We now apply Proposition 4.2.1.3. □

Remark 4.2.1.5. When $Z_G(N)$ is smooth over k and p is good for G, the dimension of the centralizer is independent of k (given the combinatorial classification of the G-orbit of N). This is apparent as the classification of nilpotent orbits and their dimensions are independent of k.

Remark 4.2.1.6. Additional analysis gives a broader criterion: $Z_G(N)$ is smooth for a connected reductive group G provided the derived group G' is simply connected, p is good for G, and there exists a G-invariant non-degenerate bilinear form on $\mathfrak{g}$ [Jan04, §2.9].

4.2.2 Centralizers in Classical Cases

For later use, we now explicitly describe the centralizer $Z_G(N)$ for symplectic and orthogonal groups over an algebraically closed field k of characteristic $p \neq 2$ and for suitable N. Let $G = \mathrm{O}_m$ or $G = \mathrm{Sp}_m$ with $m \geq 4$, acting on $M = k^m$ preserving the standard pairing φ. Choose $\epsilon = 1$ in the case of O_m and $\epsilon = -1$ in the case of Sp_m. Let σ be an admissible partition $m = m_1 + \ldots + m_r$. We will work with $(M_\sigma = \mathcal{O}^m, \varphi_\sigma, X_\sigma)$ as defined above Lemma 4.1.2.7, so $\underline{\mathrm{Aut}}(M_\sigma, \varphi_\sigma)_k \simeq G$. Write X to denote the nilpotent $(X_\sigma)_k$ with σ fixed, and M to denote $(M_\sigma)_k = k^m$. We know there exist vectors $v_1, \ldots, v_r \in M$ such that

$$v_1, Xv_1, \ldots, X^{m_1-1}v_1, v_2, Xv_2, \ldots, X^{m_r-1}v_r$$

is a basis for M. Furthermore, $X^{d_i}v_i = 0$ for $i = 1, \ldots r$, and the pairing between basis elements is given as near the start of the proof of Proposition 4.1.2.8. In particular, each v_i pairs non-trivially with only one other basis vector $X^{d_i-1}v_{i^*}$, for some $i^* \in \{1, \ldots, r\}$. (This is completely spelled out in [Jan04, §1.11].)

To understand the G-centralizer of X, we construct an associated grading of V as in [Jan04, §3.3,3.4]. This is motivated by the Jacobson-Morosov theory of $\mathfrak{sl}_2$-triples over a field of sufficiently large characteristic, but for symplectic and orthogonal groups it is constructed by hand in characteristic $p \neq 2$ below.

Remark 4.2.2.1. Every nilpotent X gives a filtration of V defined by $\mathrm{Fil}^i = \ker(X^i)$. For GL_n, this is a good filtration and is used in [CHT08] to define the minimally ramified deformation condition for GL_n. However, this filtration need not be isotropic with respect to the pairing, so we will construct a nicer grading associated to X.

Remark 4.2.2.2. The motivation for the grading to be associated to X comes from the Jacobson-Morosov theorem. Let $\mathfrak{g}$ be the Lie algebra of a reductive group G over an algebraically closed field of characteristic p, where either $p = 0$ or $p > 3(h-1)$ where h is the Coxeter number of G. An $\mathfrak{sl}_2$-*triple* is triple of non-zero elements $H, X, Y \in \mathfrak{g}$ such that

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

(They span a subalgebra isomorphic to $\mathfrak{sl}_2$). The Jacobson-Morosov theorem says every non-zero nilpotent X can be extended to a $\mathfrak{sl}_2$-triple. The advantage is that we understand the representation theory of $\mathfrak{sl}_2$ in characteristic zero: the irreducible representations are the representation on $L_d = \mathrm{Sym}^d k^2$. In positive characteristic there are differences from the characteristic zero case in dimension greater than p, but the assumption that $p > 3(h-1)$ ensures that these problems are not relevant for our purposes. Viewing the adjoint representation as a representation of $\mathfrak{sl}_2$, it decomposes as a direct sum. On a summand L_d, X and Y act as raising and lowering operators, and H has eigenvalues $d, d-2, \ldots, -d$. The one-dimensional eigenspaces of H give a grading on the Lie algebra: the ith graded piece is

$$\{g \in \mathfrak{g} : H.g = i \cdot g\}.$$

Definition 4.2.2.3. Let $M(s)$ be the span of $X^j v_i$ for all i and j such that $s = 2j + 1 - d_i$. We set $M^{(s)} = \bigoplus_{t \geq s} M(t)$, and also define $N(s)$ to be the span of $\{v_i : v_i \in M(s)\}$.

Example 4.2.2.4. Take $G = \mathrm{O}_8$ and σ to be the admissible partition $1 + 2 + 2 + 3$. There are vectors $v_1, v_2, v_3, v_4 \in k^8 = M$ such that

$$v_1, v_2, Xv_2, v_3, Xv_3, v_4, Xv_4, X^2v_4$$

form a basis for M. Then $M(-2) = \mathrm{Span}_k(v_4)$, $M(-1) = \mathrm{Span}_k(v_2, v_3)$, $M(0) = \mathrm{Span}_k(v_1, Xv_4)$, $M(1) = \mathrm{Span}_k(Xv_2, Xv_3)$, and $M(2) = \mathrm{Span}_k(X^2v_4)$. Furthermore, $N(0) = \mathrm{Span}_k(v_1)$, $N(-1) = \mathrm{Span}_k(v_2, v_3)$, and $N(-2) = \mathrm{Span}_k(v_4)$. The nilpotent X raises the degree by 2, and $N(s)$ plays the role of a space of "lowest-weight vectors" inside $M(s)$ with respect to the operator X.

We now record some elementary properties of the preceding construction; all are routine to check, and may be found in [Jan04, §3.4]. We have that $M = \bigoplus_s M(s)$, and

$$v_i \in M(-(d_i-1)), Xv_i \in M(-(d_i-1)+2), \ldots, X^{d_i-1}v_r \in M(d_r-1).$$

Furthermore, we know $XM(s) \subset M(s+2)$ and $M(s) = XM(s-2) \oplus N(s)$ for $s \leq 0$.

The dimension of $M(s)$ is $m_s(\sigma) := \{j : d_j - 1 \geq |s|\}$. The dimension of $N(s)$ equals $n_s(\sigma) := m_{s+1}(\sigma) - m_s(\sigma)$. Furthermore, the pairing φ interacts well with the grading: a computation with basis elements gives that

$$\varphi(M(s), M(t)) \neq 0 \quad \Longrightarrow \quad s + t = 0.$$

In particular, $(M^{(r)})^\perp = M^{(-r)}$.

The above grading on M corresponds to the one-parameter subgroup $\lambda : \mathbf{G}_m \to G$ for which the action of $t \in \mathbf{G}_m$ on $M(s)$ is given by scaling by t^s. The dynamic method (see [CGP15, §2.1, Proposition 2.2.9]) associates to λ a parabolic subgroup $P_G(\lambda)$ with Levi $Z_G(\lambda)$. Define C_X and U_X to be the scheme-theoretic intersections

$$C_X = Z_G(X) \cap Z_G(\lambda) = \{g \in Z_G(X) : gM(i) = M(i) \text{ for all } i\}$$
$$U_X = Z_G(X) \cap U_G(\lambda) = \{g \in Z_G(X) : (g-1)M^{(i)} \subset M^{(i+1)} \text{ for all } i\}.$$

Fact 4.2.2.5. *The group-scheme $Z_G(X)$ is a semi-direct product of C_X and the smooth connected unipotent subgroup U_X.*

This is [Jan04, Proposition 3.12]. The existence of λ and this decomposition is not specific to symplectic and orthogonal groups [Jan04, Proposition 5.10].

We finally give a concrete description of C_X. We first define a pairing on $N(s)$. Recall that the space $N(s)$ of "lowest weight vectors" in $M(s)$ has basis $\{v_i : 1 - d_i = s\}$. We define a pairing on $N(s)$ by

$$\psi_s(v, w) = \varphi(v, X^{1-s}w).$$

A direct calculation shows that ψ_s is non-degenerate and that ψ_s is symmetric if $(-1)^{1-s} = -\epsilon$ and is alternating if $(-1)^{1-s} = \epsilon$ [Jan04, §3.7].

A point of C_X preserves the grading on M, and since it commutes with "raising operator" X its action on M is determined by its action on the space $N(s)$ of "lowest weight vectors" in $M(s)$, the following fact is no surprise.

Fact 4.2.2.6. *There is an isomorphism of algebraic groups*

$$C_X \simeq \prod_{s \leq -1} \underline{\mathrm{Aut}}(N(s), \psi_s)$$

This is [Jan04, §3.8 Proposition 2, 3].

Example 4.2.2.7. Let $G = \mathrm{Sp}_m$. Unraveling when ψ_s is symmetric or alternating, we see that

$$C_X \simeq \prod_{s \leq -1; s \,\mathrm{even}} O(N(s), \psi_s) \times \prod_{s \leq -1; s \,\mathrm{odd}} \mathrm{Sp}(N(s), \psi_s).$$

The symplectic factors are connected, while the orthogonal factors have two connected components. The connected components of $Z_G(X)$ are the same as those for C_X by Fact 4.2.2.5. There are 2^t of them, where t is the number of even s for which $N(s) \neq 0$.

Example 4.2.2.8. Let $G = \mathrm{O}_m$. We likewise see that

$$C_X \simeq \prod_{s \leq -1; s \,\mathrm{odd}} O(N(s), \psi_s) \times \prod_{s \leq -1; s \,\mathrm{even}} \mathrm{Sp}(N(s), \psi_s).$$

The connected components of $Z_G(X)$ are the same as those for C_X by Fact 4.2.2.5. There are 2^t of them, where t is the number of odd s for which $N(s) \neq 0$.

Now suppose that $G = \mathrm{SO}_m$. The elements X we considered in this section are representatives for some of the nilpotent orbits of SO_m. The group C_X has the same structure as for $G = \mathrm{O}_m$, except we require that the overall determinant be 1; this has 2^{t-1} connected components. Though SO_m has more nilpotent orbits than O_m, according to Remark 4.1.2.4 their representatives are conjugate by an element of $\mathrm{O}_m(k) - \mathrm{SO}_m(k)$ to the representatives constructed in Proposition 4.1.2.8. But such a transformation is just a change of basis preserving the pairing, so our analysis in this section applies. Thus, we have an explicit description of the component group of $Z_G(X)$ in the SO_m case as well.

Remark 4.2.2.9. Assume $\sqrt{-1}, \sqrt{2} \in \mathcal{O}^\times$. By using the element $N_\sigma \in \mathfrak{g}$ constructed in Proposition 4.1.2.8 (which came with the data of a basis $\{v'_i\}$ for a free $\mathcal{O}$-module M of rank m), much of the discussion in this section continues to hold if we work over $\mathcal{O}$: in particular, there is no problem defining the grading $M(s)$ or cocharacter $\lambda : \mathbf{G}_m \to G$ over $\mathcal{O}$. Furthermore, the dynamic construction in [CGP15, §2.2] is carried out over rings.

Suppose q is a square in $\mathcal{O}^\times$. For use in the proof of Proposition 4.4.2.3, we need the existence of an element $\Phi \in G(\mathcal{O})$ such that $\mathrm{ad}_G(\Phi)N_\sigma = qN_\sigma$. If $\alpha^2 = q$, taking $\Phi = \lambda(\alpha)$ would work: Φ would scale $N_\sigma^j v_i \in M(s)$ by α^s, and N_σ increases the degree by 2.

This Φ is a version for symplectic and orthogonal groups of the diagonal matrix denoted $\Phi(\sigma, a, q)$ whose diagonal entries are increasing powers of q used in [Tay08, §2.3]. There it is checked that $\mathrm{ad}_G(\Phi(\sigma, a, q))N_\sigma = qN_\sigma$ where N_σ is the nilpotent representative in Jordan canonical form considered in Example 4.1.2.2 for the partition σ of m.

4.2.3 Checking Flatness over a Dedekind Base

We want to analyze smoothness of centralizers in the relative setting (especially over $\mathrm{Spec}\,\mathcal{O}$). If $Z_G(N_\sigma) \to \mathrm{Spec}\,\mathcal{O}$ is flat and the special and generic fibers are smooth then $Z_G(N)$ is smooth over $\mathcal{O}$. The following lemma gives a way to check that a morphism to a Dedekind scheme is flat.

Lemma 4.2.3.1. *Let $f : X \to S$ be finite type for a connected Dedekind scheme S. Then f is flat provided the following all hold:*

1. *for each $s \in S$, X_s is reduced and non-empty;*
2. *for each $s \in S$, X_s is equidimensional with dimension independent of s;*
3. *there are sections $\{\sigma_i \in X(S)\}$ to f such that for every irreducible component of a fiber above a closed point, there is a section σ_i which meets the fiber only in that component.*

Remark 4.2.3.2. This lemma is a modification of [GY03, Proposition 6.1] to allow several connected components in the fibers.

Proof. It suffices to prove the result when $S = \mathrm{Spec}(A)$ for A a discrete valuation ring with uniformizer π. Let X_η be the generic fiber and X_s the special fiber. Consider the schematic closure $\imath : X' \hookrightarrow X$ of the generic fiber. The scheme X' is flat over $\mathrm{Spec}(A)$ since flatness is equivalent to being torsion-free over a discrete valuation ring, and there is an exact sequence

$$0 \to J \to \mathcal{O}_X \to \imath_*\mathcal{O}_{X'} \to 0 \tag{4.2.3.1}$$

where J is a coherent sheaf killed by a power of π. We will show that $\imath$ is an isomorphism by analyzing the special fiber.

First, we claim that the dimension of each irreducible component on the special fiber of X' is the same as the dimension of the equidimensional $X'_\eta = X_\eta$. We will get this from flatness of X'. The generic fiber of X' is X_η, which is equidimensional and non-empty by hypothesis. Furthermore, X' is the union of the closures Z_i of the reduced irreducible components $X_{\eta,i}$ of X_η, and each Z_i is A-flat with integral η-fiber, hence integral. We just need to analyze the dimension of irreducible components of $(Z_i)_s$ when $(Z_i)_s \neq \emptyset$. Since Z_i is integral, we can apply [Mat89, Theorem 15.1, 15.5] to such Z_i to conclude that the dimension of each irreducible component of the special fiber of X' is the same as the dimension of the generic fiber.

Observe that the sections σ_i factor through the closed subscheme $X' \subset X$, as we can check this on the generic fiber since X' is A-flat. Thus, X' meets every irreducible component of X_s away from the other irreducible components of X_s, and so we would have that $|X'_s| = |X_s|$ if X'_s is equidimensional of the same dimension as the equidimensional X_s. We have shown the dimension of any irreducible component in X'_s is the same dimension as the common dimension of irreducible components of the generic fiber X_η of X'. By hypothesis, the dimension of any irreducible component of the generic fiber of X is the same as the dimension of any irreducible component of the special fiber of X. Thus the dimension of any irreducible component of X'_s is the same as the dimension of each irreducible component of X_s, giving that $|X'_s| = |X_s|$. As X_s is reduced, this forces $\imath_s : X'_s \hookrightarrow X_s$ to be an isomorphism.

Now tensoring (4.2.3.1) with the residue field of A gives an exact sequence

$$0 \to J/\pi J \to \mathcal{O}_{X,s} \to \imath_* \mathcal{O}_{X',s} \to 0$$

because $\mathcal{O}_{X'}$ is A-flat. Therefore $J/\pi J = 0$. Hence $J = \pi J = \pi^2 J = \ldots = \pi^n J = 0$ for n large, so $X = X'$ is flat over A. □

Corollary 4.2.3.3. *In the situation of the lemma, if the fibers are also smooth then X is smooth.*

Proof. For a flat morphism of finite type between Noetherian schemes, smoothness of all fibers is equivalent to smoothness of the morphism. □

4.2.4 Smooth Centralizers

We now return the case when G is a split reductive group scheme with connected fibers over a discrete valuation ring $\mathcal{O}$ with uniformizer π and residue field k of *very good* characteristic $p > 0$. Let $T \subset G$ be a fiber-wise maximal split $\mathcal{O}$-torus, and G' denote the derived group of G over $\mathcal{O}$. Denote the field of fractions of $\mathcal{O}$ by K, and set $\mathfrak{g} := \operatorname{Lie} G$. Suppose we are given $N = N_\sigma \in \mathfrak{g}$, an integral representative for the nilpotent orbit on geometric fibers corresponding to $\sigma \in \mathcal{C}$ as in (4.1.2.1). Proposition 4.1.2.8 provides such N in symplectic and orthogonal cases when $\sqrt{-1}, \sqrt{2} \in \mathcal{O}^\times$. We wish to show that the $Z_G(N)$ is smooth over $\mathcal{O}$ in many situations. An obviously necessary condition is that

$$Z_{G_K}(N_K) \text{ and } Z_{G_k}(N_k) \text{ are smooth.} \tag{4.2.4.1}$$

Remark 4.2.4.1. Some assumption on N is essential. Otherwise $N_{\overline{K}}$ and $N_{\overline{k}}$ can lie in "different" nilpotent orbits (in terms of the combinatorial characteristic-free classification of geometric orbits), and so $Z_{G_K}(N_K)$ and $Z_{G_k}(N_k)$ could have different dimensions, in which case $Z_G(N)$ cannot be $\mathcal{O}$-flat. An example of this is the element N_2 in Example 1.2.3.3.

Corollary 4.2.3.3 gives an approach to proving smoothness. It suffices to produce a finite set of elements of $Z_G(N)(\mathcal{O})$ collectively meeting each connected component of each geometric fiber of $Z_G(N)$ over $\operatorname{Spec} \mathcal{O}$ provided that (4.2.4.1) holds. Note that we may first make a local flat extension of $\mathcal{O}$, as it suffices to check flatness after such an extension. In particular, we may reduce to the case that k is *algebraically closed*.

In this section, we will establish the following result:

Proposition 4.2.4.2. *Suppose that all the irreducible factors of the root system $\Phi(G, T)$ are of classical type, and $\sqrt{-1}, \sqrt{2} \in \mathcal{O}^\times$. Construct $N \in \mathfrak{g}' = \operatorname{Lie} G'$ by decomposing $\mathfrak{g}'$ according to the irreducible components of $\Phi(G, T)$ and using the integral representatives provided by Proposition 4.1.2.8 for symplectic and orthogonal factors (types B, C, and D) and Example 4.1.2.2 for $\mathfrak{sl}_n$ (type A). Suppose that the fibers of $Z_G(N)$ over* $\operatorname{Spec} \mathcal{O}$ *are smooth. Then $Z_G(N)$ is smooth over* $\operatorname{Spec} \mathcal{O}$.

Remark 4.2.4.3. In particular, this gives smoothness for G equal to GSp_{2n} ($n \geq 2$) and GO_m ($m \geq 4$) using Proposition 4.2.1.4.

The main step of the proof of Proposition 4.2.4.2 is to establish that the scheme-theoretic centralizer $Z_G(N)$ is $\mathcal{O}$-flat. First, some preliminaries. We may and do assume that $\mathcal{O}$ is Henselian (or even complete) by scalar extension. For $N \in \mathfrak{g}$, we define

$$A(N) = \left(Z_{G_k}(N_k)/Z_{G_k}(N_k)^\circ\right)(k) = Z_{G_k}(N_k)(k)/Z_{G_k}(N_k)^\circ(k),$$

and study when the following holds:

$$\text{each element of } A(N) \text{ arises from some } s \in Z_G(N)(\mathcal{O}). \tag{4.2.4.2}$$

This is very easy for GL_n:

Example 4.2.4.4. Consider the nilpotent orbits of GL_n as in Example 4.1.1.2, with representatives given in Example 4.1.2.2. But $Z_{G_k}(N_k)$ is connected for every nilpotent orbit [Jan04, Proposition 3.10], so the identity section shows (4.2.4.2) holds.

Let $\pi : \widetilde{G'} \to G'$ be the simply connected central cover of the derived group G' over $\mathcal{O}$. As p is very good, $\widetilde{G'}$ and G' have isomorphic Lie algebras via π and $\operatorname{Lie} G'$ is a direct factor of $\operatorname{Lie} G$ with complement $\operatorname{Lie}(Z_G)$, so we may abuse notation and view N as an element of all of these Lie algebras over $\mathcal{O}$.

Let $S \subset T$ be the (split) maximal central torus in G. Consider the isogeny $S \times \widetilde{G'} \to G$. As S acts trivially on N, we see that $S \times Z_{\widetilde{G'}}(N)$ is the preimage of $Z_{G'}(N)$ under this isogeny. As p is very good for G, we obtain finite étale surjections

$$Z_{\widetilde{G'}}(N) \to Z_{G'}(N) \quad \text{and} \quad S \times Z_{\widetilde{G'}}(N) \to Z_G(N)$$

over $\mathcal{O}$.

Lemma 4.2.4.5. *The condition* (4.2.4.2) *holds for* $\widetilde{G'}$ *if and only if* (4.2.4.2) *holds for* G.

Proof. Assume $\widetilde{G'}$ satisfies (4.2.4.2). Pick a connected component of C of $Z_{G_k}(N_k)$. The preimage of C under $S \times Z_{\widetilde{G'}}(N) \to Z_G(N)$ is a union of k-fiber components of the form $S_k \times C'$ where C' is a connected component of $Z_{\widetilde{G'}_k}(N_k)$. By assumption, there exists $s \in Z_{\widetilde{G'}}(N)(\mathcal{O})$ meeting any such C'. The image of $(1, s)$ is a point of $Z_G(N)(\mathcal{O})$ meeting C.

Conversely, assume G satisfies (4.2.4.2). Pick a connected component C' of $Z_{\widetilde{G'}_k}(N_k)$. Under $S \times Z_{\widetilde{G'}}(N) \to Z_G(N)$, $S_k \times C'$ maps onto a connected component C of $Z_{G_k}(N_k)$. By assumption, there exists $s \in Z_G(N)(\mathcal{O})$ such $s_k \in C$. As k is algebraically closed, there is $s'_k \in (S \times Z_{\widetilde{G'}}(N))(k)$ lifting s_k and lying in C'. As $S \times Z_{\widetilde{G'}}(N) \to Z_G(N)$ is a finite étale cover and $\mathcal{O}$ is Henselian, there exists $s' \in (S \times Z_{\widetilde{G'}}(N))(\mathcal{O})$ lifting s and reducing to s'_k. ☐

Lemma 4.2.4.5 allows us to reduce to the case when G is semisimple and simply connected. In such a situation, $G = \prod_j G_j$ with G_j having irreducible root system (and p very good for each G_j) and likewise $T = \prod_j T_j$ for split maximal $\mathcal{O}$-tori $T_j \subset G_j$. The element N was defined by decomposing the Lie algebra, so we obtain a decomposition $N = \sum_j N_j$ such that $N_j \in \mathfrak{g}_j$ and $Z_G(N) = \prod_j Z_{G_j}(N_j)$.

Lemma 4.2.4.6. *If* $G = \prod_i G_i$ *and* (4.2.4.2) *holds for each* G_i*, it holds for* G.

Proof. The component group of a product is the product of the component groups over an algebraically closed field. ☐

Lemma 4.2.4.6 allows us to reduce to a semisimple, simply connected G with irreducible root system. We can now check types A_n, B_n, C_n, and D_n using previous work, and use Lemma 4.2.4.5 to reduce the cases that are traditionally studied:

1. For type A_n, take $G = \mathrm{GL}_n$ and use Example 4.2.4.4.

2. For type C_n with $n \geq 2$, use Example 4.2.2.7 and Proposition 4.1.2.8 for Sp_{2n}. Note that the representatives constructed are obviously integral.

3. For type B_n with $n \geq 2$ or D_n with $n \geq 4$, use Example 4.2.2.8 and Proposition 4.1.2.8 for SO_{2n+1} and SO_{2n} respectively. (Recall that $D_3 = A_3$ and $D_2 = A_1 \times A_1$.)

We now prove Proposition 4.2.4.2.

Proof. We are given that $Z_{G_k}(N_k)$ and $Z_{G_K}(N_K)$ are smooth. By the classification of nilpotent orbits over algebraically closed fields, the dimension of the orbit only depends on the combinatorial classification for the orbit in very good characteristic and in characteristic 0, so these fibers are equidimensional of the same dimension. By Corollary 4.2.3.3, it suffices to find $s \in Z_G(N)(\mathcal{O})$ meeting each desired connected component of $Z_{G_k}(N_k)$.

Using Lemmas 4.2.4.5 and 4.2.4.6, it suffices to do so for the irreducible root systems A_n, B_n $(n \geq 2)$, C_n $(n \geq 2)$, and D_n $(n \geq 4)$. We have done so above. ☐

The above argument worked for classical groups, which is our main application. However, we will briefly discuss some possible approaches to producing a finite set of $\mathcal{O}$-points collectively meeting all components of the geometric special fiber of a centralizer in general. Let G be a split reductive $\mathcal{O}$-group with split maximal $\mathcal{O}$-torus T, and let $N = N_\sigma \in \mathfrak{g}_{\mathcal{O}}$ be the element constructed in terms of root data as in Remark 4.1.2.1. Lemmas 4.2.4.5 and 4.2.4.6 show that to make "enough" points in $Z_G(N)(\mathcal{O})$ we can reduce to the simple and simply connected case if we so choose.

As all split reductive groups descend as such to $\mathbf{Z}$, and the description of N in terms of the root datum shows it is compatibly defined over $\mathbf{Z}$, we may assume that the residue field k is $\overline{\mathbf{F}}_p$ for the purposes of checking (4.2.4.2).

Proposition 4.2.4.7. *Suppose G is semisimple and simply connected and $\mathcal{O}$ is Henselian. For any $s_k \in Z_{G_k}(N)(k) \cap T(k)$, there exists $s \in Z_G(N)(\mathcal{O})$ lifting s_k.*

Remark 4.2.4.8. Before proving Proposition 4.2.4.7, we record a general observation. A representative $s_k \in Z_{G_k}(N_k)$ for a coset of $A(N)$ may always be chosen to be semisimple. A proof for the analogous assertion for centralizers of unipotent elements in G is given in [MS03, Corollary 13]; the proof easily adapts to nilpotents in the Lie algebra, or we can invoke the Springer isomorphism to translate between unipotents and nilpotents in very good characteristic. However, it is not always the case that such a component group representative can be chosen in a *single* maximal torus T as s_k varies. Indeed, if it could then the component group $A(N)$ would be commutative since T is commutative, but some nilpotent orbits in exceptional groups have centralizer with a non-abelian component group. When all representatives can be chosen in a single T, then by Proposition 4.2.4.7 we could verify (4.2.4.2).

Proof. Let $\Phi = \Phi(G,T)$ and Δ be a basis for a set Φ^+ of positive roots. By hypothesis, we may represent N as $N = \sum_{\beta \in J} N_\beta$ where N_β is an $\mathcal{O}$-basis for the root line $\mathfrak{g}_\beta$ and $J \subset \Phi^+$. As G is simply connected, the simple coroots span the cocharacter lattice, so we may write

$$s_k = \prod_{\alpha \in \Delta} \alpha^\vee(t_\alpha)$$

for some $t_\alpha \in \mathbf{G}_m(k) = k^\times$. As $s_k \in Z_{G_k}(N_k)$, the action of s_k on N_β is trivial for each $\beta \in J$, so for all $\beta \in J$

$$\prod_\alpha t_\alpha^{\langle \beta, \alpha^\vee \rangle} = 1.$$

As we have used the existence of Chevalley groups over $\mathbf{Z}$ to reduce to the case $k = \overline{\mathbf{F}}_p$, all of the t_α are of *finite* order. Thus we pick a root of unity $t_k \in k^\times$ such that each t_α is a power of t_k. Let the order of t_k be m, which we know is prime to p. Choose n'_α so that $t_k^{n'_\alpha} = t_\alpha$. Then we know for all $\beta \in J$

$$t_k^{\sum_\alpha n'_\alpha \langle \beta, \alpha^\vee \rangle} = 1.$$

In other words, $\sum_\alpha n'_\alpha \langle \beta, \alpha^\vee \rangle \equiv 0 \pmod{m}$.

Now pick a lift $t \in \mathcal{O}^\times$ of t_k that is an mth root of unity (as we may do since $p \nmid m$ and $\mathcal{O}$ is Henselian). The element

$$s := \prod_{\alpha \in \Delta} \alpha^\vee(t)^{n'_\alpha} \in T(\mathcal{O})$$

also lies in $Z_G(N)(\mathcal{O})$ because for any $\beta \in J$

$$t^{\sum_\alpha n'_\alpha \langle \beta, \alpha^\vee \rangle} = 1$$

(as this mth root of unity in $\mathcal{O}^\times$ has trivial reduction in $k^\times$). This says that s acts trivially on N_β for any $\beta \in J$. ∎

We now review a description of the component groups $A(N) = Z_{G_k}(N_k)/Z_{G_k}(N_k)^\circ$ over an algebraically closed field k and sketch an approach that reduces the question of whether (4.2.4.2) holds for the exceptional groups to extensive case-checking.

For a connected reductive group H of over an algebraically closed field F of very good characteristic, there is a classification of the possibilities of the component group of a centralizer $Z_H(X)$ for nilpotent $X \in \mathrm{Lie}(H)(F)$. The classification uses the notion of a *pseudo-Levi subgroup*, which is a subgroup of the form $Z_H(s)^\circ$ for a semisimple $s \in H(F)$. This classification will be in terms of triples $(L, sZ^\circ(F), X)$ where L is a pseudo-Levi subgroup of H, Z is the center of L, $sZ^\circ(F)$ is a coset of $Z^\circ(F)$ such that $L = Z_H(sZ^\circ(F))^\circ$, and X is a distinguished unipotent in $\mathrm{Lie}(L)$ (recall that being distinguished means that every maximal torus contained in $Z_H(X)$ is contained in the center of H).

Fact 4.2.4.9. *There is a bijection between $H(F)$-conjugacy classes of triples $(L, sZ^\circ(F), X)$ as above and $H(F)$-conjugacy classes of pairs (X, C) where X is a distinguished nilpotent in* $\mathrm{Lie}(H)$ *and C is a conjugacy class of $(Z_H(X)/Z_H(X)^\circ)(F)$, taking the class of $(L, sZ^\circ(F), X)$ to the class of $(X, [s])$.*

This is the main theorem of [MS03]. There it is stated for unipotent elements of $H(F)$, but in very good characteristic the Springer isomorphism identifies the unipotent and nilpotent varieties. For simple groups of adjoint type, an alternate proof is given in [Pre03, Theorem 2].

Example 4.2.4.10. Let us give an explicit construction of pseudo-Levi subgroups when the root system of H is irreducible. Pick a maximal torus $T \subset H$, and let $\Phi = \Phi(H, T)$ be the root system. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be a basis of a system of positive roots, and $I = \{1, 2, \ldots, r\}$. Let $\widetilde{\alpha} = \sum_{i \in I} n_i \alpha_i$ be the highest root. Define $\alpha_0 = -\widetilde{\alpha}$, $n_0 = 1$, and $I_0 = I \cup \{0\}$. For a proper subset $J \subset I_0$, let Φ_J the set of all roots of the form $\sum_{i \in J} a_i \alpha_i$. The subgroup L_J of H generated by T and the root groups U_α for $\alpha \in \Phi_J$ is a pseudo-Levi subgroup, and all pseudo-Levi subgroups are conjugate to one of this form [MS03, Propositions 30, 32].

Consider the case $H = \mathrm{GL}_4$ with diagonal torus T and upper triangular Borel subgroup B. The pair (B, T) determines a basis of positive simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$. The other positive roots are $\alpha_1 + \alpha_2, \alpha_1 + \alpha_3$, and the highest root $\widetilde{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$. Suppose $J \subset \{1, 2, 3\}$. Then L_J is a Levi factor of a parabolic subgroup of H. For example, if $J = \{1\}$ then L_J is generated by T, U_{α_1} and $U_{-\alpha_1}$; this L_J is the identity component of the centralizer of a diagonal element of the form

$$s = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$

with pairwise distinct $\lambda_1, \lambda_2, \lambda_3 \in k^\times$, and is *also* a Levi factor of the parabolic subgroup of block upper triangular matrices with blocks of size 2, 1 and 1. A distinguished nilpotent element in the Lie algebra of L_J is

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the centralizer of X is connected (as is always the case for GL_n). The pair (L_J, X) corresponds to the pair $(X, [s])$, where s lies in the identity component of the centralizer of X

If $0 \in J$, the pseudo-Levi L_J may be less familiar. Consider $J' = \{0, 1\}$. Then $L_{J'}$ is generated by T, U_{α_1}, $U_{\alpha_1+\alpha_2+\alpha_3}$, $U_{\alpha_2+\alpha_3}$, $U_{-\alpha_1-\alpha_2-\alpha_3}$, $U_{-\alpha_1}$, and $U_{-\alpha_2-\alpha_3}$, and it is also the identity component of the centralizer of a diagonal element s' of the form

$$s' = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

with pairwise distinct $\lambda_1, \lambda_2 \in k^\times$. Alternately, $L_{J'}$ is a Levi factor of the parabolic subgroup determined by the flag $0 \subset \mathrm{span}(e_1, e_2, e_4) \subset F^4$ (where $\{e_1, e_2, e_3, e_4\}$ is the standard basis of F^4).

Remark 4.2.4.11. In our example, when $0 \in J$ we saw that L_J is a Levi factor of a parabolic subgroup of H. This is not true in general. In fact, a pseudo-Levi subgroup is a Levi factor of a parabolic subgroup if and only if the pair (L, X) corresponds to $(X, [1])$ [Pre03, Theorem 3.7]. But $Z_H(X)$ has more than one connected component in general.

We now return to the relative situation over $\mathcal{O}$, and remember that we have extended $(\mathcal{O}, k)$ so that k is algebraically closed.

Lemma 4.2.4.12. *Suppose that for every conjugacy class of $A(N)$, there exists an element $s \in Z_G(N)(\mathcal{O})$ such that $sZ_{G_k}(N_k)^\circ$ lies in that conjugacy class. Then for every coset in $A(N)$ there exists an element $s \in Z_G(N)(\mathcal{O})$ representing that coset.*

Proof. Consider the subset

$$\Sigma = \{sZ_{G_k}(N_k)^\circ : s \in Z_G(N)(\mathcal{O})\} \subset A(N)$$

It is certainly a subgroup, and it meets every conjugacy class of the finite group $A(N)$. Consider the decomposition $A(N) = \cup_{g \in A(N)/\Sigma}\, g\Sigma g^{-1}$. Each $g\Sigma g^{-1}$ contains the identity, so by comparing sizes we see that $A(N) = \Sigma$. □

Thus it would suffice to show that for each triple $(L_k, s_k Z_k^\circ, N_k)$ produced in Fact 4.2.4.9 (applied to G_k over k in the role of H and F), s_k is the reduction of an element $s \in G(\mathcal{O})$ which acts trivially on N. We may conjugate so that $s_k \in T(k)$, in which case $T_k \subset L_k$. Doing so changes N_k, so we no longer have an explicit description of N in terms of root data. Without such a description, it is not clear how to lift s_k using cocharacters as in Proposition 4.2.4.7.

We may conjugate so that L_k is of the form L_J which contains T_k. The k-group L_J is reductive [MS03, Lemma 14], and p is good for L_J [MS03, Proposition 16]. The nilpotent N_k lies in a nilpotent orbit for L_J, so as in Remark 4.1.2.1 it is $L_J(k)$-conjugate to some nilpotent $\overline{N}'$ which can be expressed in terms of the roots of $\Phi(L_J, T_k)$. In particular, $\overline{N}'$ has a natural lift N' to $G(\mathcal{O})$. However, it is not clear that N' and the original N are $G(\mathcal{O})$-conjugate. Both are described in terms of roots of $\Phi(G, T)$, so for the *finite* number of nilpotent orbits in exceptional groups this assertion could be attacked by (unpleasant) case-checking. As s_k belongs to T_k and N' is described in terms of $\Phi(G, T)$, Proposition 4.2.4.7 would then lift s_k. Of course, a uniform approach would be preferable.

Remark 4.2.4.13. McNinch analyzes the centralizer of an "equidimensional nilpotent" in [McN08]. An *equidimensional nilpotent* is an element $N \in \mathfrak{g}$ such that N_K is nilpotent and the dimension of the special and generic fibers of $Z_G(N)$ are the same. [McN08, §5.2] claims that such $Z_G(N)$ are $\mathcal{O}$-smooth because the fibers are smooth of the same dimension. This deduction is *incorrect*: it relies on [McN08, 2.3.2] which uses the wrong definition of an equidimensional morphism and thereby incorrectly applies [SGA1, Exp. II, Prop 2.3].

According to [SGA1, Exp. II, Prop 2.3] (or see [EGAIV$_3$, §13.3, 14.4.6, 15.2.3]), for a Noetherian scheme Y, a morphism $f : X \to Y$ locally of finite type, and points $x \in X$ and $y = f(x)$ with $\mathscr{O}_y$ normal, f is smooth at x if and only if f is equidimensional at x and $f^{-1}(y)$ is smooth over $k(y)$ at x. But by definition in [EGAIV$_3$, 13.3.2], an *equidimensional* morphism is more than just a morphism all of whose fibers are of the same dimension (the condition checked in [McN08, 2.3.2]): a locally finite type morphism f is called equidimensional of dimension d at $x \in X$ when there exists an open neighborhood U of x such that for every irreducible component Z of U through x, $f(Z)$ is dense in some irreducible component of Y containing y and for all $x' \in U$ the fiber $f^{-1}(f(x')) \cap U$ has all irreducible components of dimension d.

This is much stronger than the fibers simply being of the same dimension. To see the importance of the extra conditions, consider a discrete valuation ring $\mathcal{O}$ with field of fractions K and residue field k, and the morphism from X, the disjoint union of $\operatorname{Spec} K$ and $\operatorname{Spec} k$, to $Y = \operatorname{Spec} \mathcal{O}$. The fibers are of the same dimension (zero) and smooth but the morphism is not flat. This morphism is also not equidimensional at $\operatorname{Spec} k$: the only irreducible component of X containing $\operatorname{Spec} k$ is the point itself, with image the closed point of $\operatorname{Spec} \mathcal{O}$. This is not dense in $\operatorname{Spec} \mathcal{O}$, the only irreducible component of the only open set containing the closed point of $\operatorname{Spec} \mathcal{O}$.

The smoothness of centralizers of an equidimensional pure nilpotent is important to proving the main results of [McN08]. In particular, the results in §6 and §7 in [McN08] crucially rely on the smoothness of the centralizers of such nilpotents, leaving a gap in the proof of Theorem B in [McN08] concerning the component group of centralizers. The method we have discussed here reverses this, understanding the geometric component group well enough to *produce* sufficiently many $\mathcal{O}$-valued points in order to deduce smoothness of the centralizer in classical cases in very good characteristic via Lemma 4.2.3.1.

4.3 Deformations of Nilpotent Elements

As before, let $\mathcal{O}$ be a discrete valuation ring with residue field k of characteristic $p > 0$, and let G be a split reductive group scheme over $\mathcal{O}$ (with connected fibers) such that p is very good for G. Let $\mathfrak{g} = \operatorname{Lie} G$. For a nilpotent element $\overline{N} \in \mathfrak{g}_k$, we define the notion of a *pure nilpotent* lift of $\overline{N}$ in $\mathfrak{g}$ and study the space of such lifts.

4.3.1 Pure Nilpotent Lifts

Let $\overline{N} \in \mathfrak{g}_k$ be a nilpotent with Bala-Carter data $\sigma \in C$. Suppose there exists $N_\sigma \in \mathfrak{g}$ lifting $\overline{N}$ such that $(N_\sigma)_{\overline{K}} \in O_{\overline{K},\sigma}$ and $Z_G(N_\sigma)$ is smooth over $\mathcal{O}$. We will define the notion of a "pure nilpotent" $N \in \mathfrak{g}$ lifting $\overline{N}$ with N_K nilpotent.

Remark 4.3.1.1. For classical groups, Proposition 4.2.4.2 shows that for any nilpotent $\overline{N} \in \mathfrak{g}_k$, there exists $N'_\sigma \in \mathfrak{g}$ such that $(N'_\sigma)_{\overline{k}} \in O_{\overline{k},\sigma}$, $Z_G(N'_\sigma)$ is $\mathcal{O}$-smooth, and such that $(N'_\sigma)_k$ and $\overline{N}$ are $G(\overline{k})$-conjugate. Thus $(N'_\sigma)_k$ and $\overline{N}$ are conjugate by $\overline{g} \in G(k')$ for some finite extension k'/k. Lift $\overline{g}$ to an element $g \in G(\mathcal{O}')$ for a Henselian discrete valuation ring local over $\mathcal{O}$ and having residue field k'. The element $gN'_\sigma g^{-1} \in \mathfrak{g}_{\mathcal{O}'}$ reduces to $\overline{N}_{k'}$ and has the required properties. So the above hypothesis is satisfied for classical groups after a finite flat local extension of $\mathcal{O}$.

Definition 4.3.1.2. Define the functor $\mathrm{Nil}_{\overline{N}} : \widehat{\mathcal{C}}_{\mathcal{O}} \to \mathrm{Sets}$ by

$$\mathrm{Nil}_{\overline{N}}(R) = \{N \in \mathfrak{g}_R |\, \mathrm{Ad}_G(g)(N_\sigma) = N \,\text{for some}\, g \in \widehat{G}(R), N_k = \overline{N}\}.$$

Call these $N \in \mathrm{Nil}_{\overline{N}}(R)$ the *pure nilpotents* lifting $\overline{N}$.

This is obviously a subfunctor of the formal neighborhood of $\overline{N}$ in the affine space $\underline{\mathfrak{g}}$ over $\mathcal{O}$ attached to $\mathfrak{g}$. The key to analyzing $\mathrm{Nil}_{\overline{N}}$ is that $Z_{G_R}(N)$ is smooth over R since $Z_G(N_\sigma)$ is $\mathcal{O}$-smooth and $(N_\sigma)_R$ is in the G-orbit of N. To ease notation below, we shall write gNg^{-1} rather than $\mathrm{Ad}_G(g)(N)$ for $g \in \widehat{G}(R)$.

Lemma 4.3.1.3. *The functor* $\mathrm{Nil}_{\overline{N}}$ *is pro-representable.*

Proof. We will use Schlessinger's criterion to check pro-representability. As $\mathrm{Nil}_{\overline{N}}$ is a subfunctor of the formal neighborhood of the scheme $\underline{\mathfrak{g}}$ at $\overline{N}$, the only condition to check is the analogue of Definition 2.2.2.7(2): given a Cartesian diagram in $\mathcal{C}_{\mathcal{O}}$

$$\begin{array}{ccc} R_1 \times_{R_0} R_2 & \xrightarrow{\pi_2} & R_2 \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow \\ R_1 & \longrightarrow & R_0 \end{array}$$

and $N_i \in \mathrm{Nil}_{\overline{N}}(R_i)$ such that N_1 and N_2 reduce to N_0, we want to check that $N_1 \times N_2 \in \mathrm{Nil}_{\overline{N}}(R_1 \times_{R_0} R_2)$. By definition, there exists $g_1 \in \widehat{G}(R_1)$ and $g_2 \in \widehat{G}(R_2)$ such that $N_1 = g_1 N_\sigma g_1^{-1}$ and $N_2 = g_2 N_\sigma g_2^{-1}$. Consider the element $g_1 g_2^{-1} \in \widehat{G}(R_0)$. Observe that

$$g_1 g_2^{-1} N_\sigma g_2 g_1^{-1} = g_1 N_0 g_1^{-1} = N_\sigma \in \mathfrak{g}_{R_0}.$$

In particular, $g_1 g_2^{-1} \in Z_G(N_\sigma)(R_0)$. The extension $R_2 \to R_0$ has nilpotent kernel, so as $Z_G(N_\sigma)$ is smooth over $\mathcal{O}$ there exists $h \in Z_G(N_\sigma)(R_2)$ lifting $g_1 g_2^{-1}$. The element

$$(g_1, hg_2) \in R_1 \times_{R_0} R_2$$

conjugates $N_1 \times N_2$ to N_σ. Hence $N_1 \times N_2 \in \mathrm{Nil}_{\overline{N}}(R_1 \times_{R_0} R_2)$. □

Lemma 4.3.1.4. *The functor* $\mathrm{Nil}_{\overline{N}}$ *is formally smooth, in the sense that for a small surjection* $R_2 \to R_1$ *of coefficient* $\mathcal{O}$*-algebras the map*

$$\mathrm{Nil}_{\overline{N}}(R_2) \to \mathrm{Nil}_{\overline{N}}(R_1)$$

is surjective. Moreover, $\mathrm{Nil}_{\overline{N}}$ *has relative dimension* $\dim G_k - \dim Z_{G_k}(N_k)$ *over* $\mathcal{O}$.

Proof. Given $N \in \mathrm{Nil}_{\overline{N}}(R_1)$, there exists $g \in \widehat{G}(R_1)$ such that $gNg^{-1} = N_\sigma$. As G is smooth over $\mathcal{O}$, we may find $g' \in \widehat{G}(R_2)$ lifting g. Then $(g')^{-1} N_\sigma g'$ is a lift of N to R_2. From its definition, the tangent space to $\mathrm{Nil}_{\overline{N}}$ is $\mathfrak{g}_k / \mathfrak{z}_{\mathfrak{g}}(N_k)$, so the formally smooth $\mathrm{Nil}_{\overline{N}}$ has relative dimension $\dim G_k - \dim Z_{G_k}(N_k)$ since $Z_G(N)$ is $\mathcal{O}$-smooth. ☐

Now suppose that A is a complete local Noetherian $\mathcal{O}$-algebra with residue field k.

Lemma 4.3.1.5. *The inverse limit* $\varprojlim \mathrm{Nil}_{\overline{N}}(A/\mathfrak{m}_A^n)$ *equals* $\{N \in \mathfrak{g}_A : N = gN_\sigma g^{-1}$ *for some* $g \in G(A)\}$.

Proof. It is immediate that the second is a subset of the first. On the other hand, suppose we had compatible elements $N_i \in \mathrm{Nil}_{\overline{N}}(A/\mathfrak{m}_A^i)$ such that N_i is $\widehat{G}(A/\mathfrak{m}_A^i)$-conjugate to N_σ.

By induction, we will show there exists $g_i \in \widehat{G}(A/\mathfrak{m}_A^i)$ such that $N_i = g_i N_\sigma g_i^{-1}$ and g_i reduces to g_{i-1}. The base case $i = 1$ is just the assertion that $(N_\sigma)_k$ equals N_1. Given $g_i \in \widehat{G}(A/\mathfrak{m}_A^i)$, we know there is some element $g'_{i+1} \in \widehat{G}(A/\mathfrak{m}_A^{i+1})$ such that $N_i = g'_{i+1} N_\sigma (g'_{i+1})^{-1}$. The element $(g'_{i+1})^{-1} g_i$ lies in $Z_G(N_\sigma)(A/\mathfrak{m}_A^i)$. As $Z_G(N_\sigma)$ is smooth over $\mathcal{O}$, we may lift to produce an element $\widetilde{g} \in Z_G(N_\sigma)(A/\mathfrak{m}_A^{i+1})$ for which $N_i = g'_{i+1}\widetilde{g} N_\sigma (g'_{i+1}\widetilde{g})^{-1}$ and such that $g'_{i+1}\widetilde{g}$ reduces to $g_i \in \widehat{G}(A/\mathfrak{m}_A^i)$. This completes the induction.

Finally let $g \in \widehat{G}(A)$ be the limit of the g_i and observe $gN_\sigma g^{-1}$ is the limit of the N_i. ☐

Remark 4.3.1.6. If we had defined $\mathrm{Nil}_{\overline{N}}$ on the larger category $\widehat{\mathcal{C}}_{\mathcal{O}}$ in the obvious way, Lemma 4.3.1.5 would say that $\mathrm{Nil}_{\overline{N}}$ is continuous.

Remark 4.3.1.7. One could define a scheme-theoretic "nilpotent cone" over $\mathcal{O}$ as the vanishing locus of the ideal of non-constant homogeneous G-invariant polynomials on $\mathfrak{g}$. The arguments in this section could be rephrased as constructing a formal scheme of pure nilpotents inside the formal neighborhood of $\overline{N}$ in $\mathfrak{g}$. A natural question is whether there is a broader notion of pure nilpotents that gives a locally closed subscheme of the scheme-theoretic nilpotent cone. For instance, for $N, N' \in \mathfrak{g}$, if their images in $\mathfrak{g}_K$ and $\mathfrak{g}_k$ are nilpotent in orbits with the same combinatorial parameters, are N and N' conjugate under G over a discrete valuation ring local over $\mathcal{O}$?

When $G = \mathrm{GL}_n$, this has been explored by Taylor in the course of constructing local deformation conditions [Tay08, Lemma 2.5]. The method uses the explicit description of the orbit closures in Example 4.1.1.4 to define an analogue over $\mathcal{O}$. It would be interesting to find a way to do the same for a general split connected reductive group.

4.4 The Minimally Ramified Deformation Condition for Tamely Ramified Representations

In this section, we will generalize the minimally ramified deformation condition introduced in [CHT08, §2.4.4] for GL_n to symplectic and orthogonal groups. We also explain why another more immediate notion based on parabolic subgroups, giving the same deformation condition for GL_n, is *not liftable* in general (even for GSp_4).

Let G be either GSp_m or GO_m (or GL_m to recover the results of [CHT08, §2.4.4]) over the ring of integers $\mathcal{O}$ in a p-adic field with uniformizer π and residue field k of characteristic $p > 0$ with $m \geq 4$. As always, we assume that p is very good for G_k (i.e. $p \neq 2$). Let $\mathfrak{g} = \mathrm{Lie}(G)$.

4.4.1 Passing between Unipotents and Nilpotents

As in §4.3.1, we work with a pure nilpotent $N_\sigma \in \mathfrak{g}$ for which $Z_G(N_\sigma)$ is $\mathcal{O}$-smooth, $(N_\sigma)_{\overline{K}} \in O_{\overline{K},\sigma}$, and $(N_\sigma)_{\overline{k}} \in O_{\overline{k},\sigma}$. Define $\overline{N} := (N_\sigma)_k$. We studied deformations of $\overline{N}$ in §4.3.1, but will ultimately want to analyze deformations of Galois representations which take on unipotent values at certain elements of a local Galois group. Thus, we need a way to pass between unipotent and nilpotent elements. For classical groups, we can use a truncated version of the exponential and logarithm maps:

Fact 4.4.1.1. *Suppose that $p \geq m$ and that R is an $\mathcal{O}$-algebra. If $A \in \mathrm{Mat}_m(R)$ has characteristic polynomial x^m then*

$$\exp(A) := 1 + A + A^2/2 + \ldots + A^{m-1}/(m-1)!$$

has characteristic polynomial $(x-1)^m$. If $B \in \mathrm{Mat}_m(R)$ has characteristic polynomial $(x-1)^m$ then

$$\log(B) := (B-1) - (B-1)^2/2 + \ldots + (-1)^m(B-1)^{m-1}/(m-1)$$

has characteristic polynomial x^m. Furthermore for $C \in \mathrm{GL}_m(R)$ we have $\exp(CAC^{-1}) = C\exp(A)C^{-1}$, $\log(CBC^{-1}) = C\log(B)C^{-1}$, $\log(\exp(A)) = A$, $\exp(\log(B)) = B$, $\exp(qA) = \exp(A)^q$, and $\log(B^q) = q\log(B)$ for any integer q.

This is [Tay08, Lemma 2.4]. The key idea is that because all the higher powers of A and $B-1$ vanish and all of the denominators appearing are invertible as $p \geq m$, we can deduce these facts from results about the exponential and logarithm in characteristic zero.

Suppose J is the matrix for a perfect symmetric or alternating pairing over R.

Corollary 4.4.1.2. *For A and B as in Fact 4.4.1.1 with $\exp(A) = B$, $A^T J + JA = 0$ if and only if $B^T JB = J$.*

Proof. Directly from the definitions we see that $\exp(A^T) = \exp(A)^T$. Observe that $\exp(JAJ^{-1}) = JBJ^{-1}$ and $\exp(-A^T) = (B^T)^{-1}$. Thus $JAJ^{-1} = -A^T$ if and only if $(B^T)^{-1} = JBJ^{-1}$. $\square$

We shall use this exponential map to convert pure nilpotents into unipotent elements. Let R be a coefficient ring over $\mathcal{O}$. By Definition 4.3.1.2, any pure nilpotent $N \in \mathrm{Nil}_{\overline{N}}(R)$ is $G(R)$-conjugate to N_σ, so it has characteristic polynomial x^m. Denoting the derived group of G by G', any element of $\mathfrak{g}$ nilpotent in $\mathfrak{g}_k$ lies in $\mathfrak{g}' = \mathrm{Lie}\, G'$ (if $G = \mathrm{GSp}_m$ for example, this means that $NJ + JN = 0$, not just $NJ + JN = \lambda J$ for some $\lambda \in \mathcal{O}$). Thus, Corollary 4.4.1.2 shows that $\exp(N) \in G(R)$. This gives an exponential map

$$\exp : \mathrm{Nil}_{\overline{N}} \to G \tag{4.4.1.1}$$

such that for $g \in \widehat{G}(R)$, $N \in \mathrm{Nil}_{\overline{N}}(R)$, and $q \in \mathbf{Z}$ we have $\exp(qN) = \exp(N)^q$ and $g\exp(N)g^{-1} = \exp(\mathrm{Ad}_G(g)N)$.

Remark 4.4.1.3. This is a realization over $\mathcal{O}$ of a special case of the Springer isomorphism identifying the nilpotent and unipotent varieties in very good characteristic. For later purposes, we will use that the identification to be compatible with the multiplication in the sense that $\exp(qA) = \exp(A)^q$. In the case of GL_m, a Springer isomorphism that works in any characteristic is given by $X \to 1 + X$ for nilpotent X, but this is not compatible with multiplication.

Remark 4.4.1.4. Let G be a split reductive $\mathcal{O}$-group with U the unipotent radical of a parabolic $\mathcal{O}$-subgroup P of G. Let r be the nilpotence class of U_K (the smallest integer for which $x^r = 1$ for all $x \in U_K$) and suppose that $p > r$. According to [Sei00, §5] (following [Ser94]), there is an exponential isomorphism $\exp_U : \mathrm{Lie}\, U \to U$ defined over $\mathcal{O}$. Making $\mathrm{Lie}\, U$ into an $\mathcal{O}$-group using the Hausdorff formula, this exponential is a map of $\mathcal{O}$-groups, so $\exp_U(qN) = \exp_U(N)^q$ for $N \in \mathrm{Lie}\, U$ and $q \in \mathbf{Z}$.

When p is larger than the Coxeter number of G, this provides an approach to converting pure nilpotents $N \in \mathrm{Nil}_{\overline{N}}(R)$ into elements of $G(R)$. Having to select a U in order to define the exponential map adds complexity; as we only need results for the classical groups, we do not purse this here.

4.4.2 Minimally Ramified Deformations

Assume the residue field k of $\mathcal{O}$ is finite of characteristic p, with $\mathcal{O}$ complete. Now suppose L is a finite extension of $\mathbf{Q}_\ell$ (with $\ell \neq p$), and denote its absolute Galois group by Γ_L. Consider a representation $\overline{\rho} : \Gamma_L \to G(k)$. We wish to define a (large) smooth deformation condition for $\overline{\rho}$ generalizing the minimally ramified deformation condition for GL_n defined in [CHT08, §2.4.4]. In this section we do so for a special class of tamely ramified representations. This requires making an étale local extension of $\mathcal{O}$, which will be harmless for our purposes.

Recall that Γ_L^{t}, the Galois group of the maximal tamely ramified extension of L, is isomorphic to the semi-direct product

$$\widehat{\mathbf{Z}} \ltimes \prod_{p' \neq \ell} \mathbf{Z}_{p'}$$

where $\widehat{\mathbf{Z}}$ is generated by a Frobenius ϕ and the conjugation action by ϕ on the direct product is given by the cyclotomic character. We consider representations of Γ_L^{t} which factor through the quotient $\widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$. Picking a topological generator τ for $\mathbf{Z}_p$, the action is explicitly given by

$$\phi \tau \phi^{-1} = q\tau$$

where q is the size of the residue field of L. Note q is a power of ℓ, so it is relatively prime to p. This leads us to study representations of the group $T_q = \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$.

Let $\overline{\rho} : T_q \to G(k)$ be a representation. Informally, the deformation condition will be lifts $\rho : T_q \to G(R)$ such that $\rho(\tau)$ lies in the "same" unipotent orbit as $\overline{\rho}(\tau)$. To make this meaningful over an infinitesimal thickening of k, we shall use the notion of pure nilpotents as in Definition 4.3.1.2 since unipotence and unipotent orbits are not good notions when not over a field.

First, let us check that $\overline{\rho}(\tau) \in G(k)$ actually is unipotent. This element decomposes as a commuting product of semi-simple and unipotent elements of $G(k)$. The order of a semi-simple element in $G(k)$ is prime to p, while by continuity there is an $r \geq 0$ such that $\tau^{p^r} \in \ker(\overline{\rho})$. Thus $\overline{\rho}(\tau)$ is unipotent. In particular, $\log(\overline{\rho}(\tau)) = \overline{N}$ is nilpotent, and by Remark 4.3.1.1 after making an étale local extension of $\mathcal{O}$ we may assume that there exists a pure nilpotent $N_\sigma \in \mathfrak{g}$ lifting $\overline{N}$ for which $Z_G(N_\sigma)$ is smooth. Making a further extension if necessary, we may also assume that the unit $q \in \mathcal{O}^\times$ is a square. We obtain an exponential map $\exp : \mathrm{Nil}_{\overline{N}} \to G$ as in (4.4.1.1).

Definition 4.4.2.1. For a coefficient ring R over $\mathcal{O}$, a continuous lift $\rho : T_q \to G(R)$ of $\overline{\rho}$ is *minimally ramified* if $\rho(\tau) = \exp(N)$ for some $N \in \mathrm{Nil}_{\overline{N}}(R)$.

Example 4.4.2.2. Take $G = \mathrm{GL}_n$. Then $X \mapsto 1_n + X$ gives an identification of nilpotents and unipotents. Up to conjugacy, over algebraically closed fields parabolic subgroups correspond to partitions of n and every nilpotent orbit is the Richardson orbit of such a parabolic. Let $\overline{\rho}(\tau) - 1_n =: \overline{N}$ correspond to the partition $\sigma = n_1 + n_2 + \ldots + n_r$. By Example 4.1.2.2, the lift N_σ of $\overline{N}$ is conjugate to a block nilpotent matrix with blocks of size $n_1, n_2, \ldots, n_r$. The points $N \in \mathrm{Nil}_{\overline{N}}(R)$ are the $\widehat{G}(R)$-conjugates of N_σ. It is clear (since $p > n$) that if $\rho(\tau) \in \mathrm{Nil}_{\overline{N}}(R)$ then

$$\ker(\rho(\tau) - 1_n)^i \otimes_R k \to \ker(\overline{\rho}(\tau) - 1_n)^i \tag{4.4.2.1}$$

is an isomorphism for all i. Conversely, repeated applications of [CHT08, Lemma 2.4.15] show that any $\rho(\tau)$ satisfying this collection of isomorphism conditions is $\widehat{G}(R)$-conjugate to N_σ. Thus the minimally ramified deformation condition for GL_n defined in [CHT08] agrees with our definition. Note that the identification $X \mapsto 1_n + X$ does not satisfy $qX \to (1+X)^q$, so it will not work in our argument. The proof of [CHT08, Lemma 2.4.19] uses a different method for which this non-homomorphic identification suffices.

Our goal is to show that the functor of minimally ramified lifts is pro-representable over $\mathcal{O}$ and that the representing object $R_{\overline{\rho}}^{\mathrm{m.r.}\,\square}$ is formally smooth over $\mathcal{O}$.

Proposition 4.4.2.3. *Under our assumptions on G, the local deformation ring $R_{\overline{\rho}}^{\mathrm{m.r.}\,\square}$ is formally smooth over $\mathcal{O}$ of relative dimension* $\dim G_k$.

Proof. Let $\overline{\Phi} = \overline{\rho}(\phi) \in G(k)$ and let $\widehat{G}_{\overline{\Phi}}$ be the formal completion of G at $\overline{\Phi}$. Using the relation

$$\overline{\rho}(\phi)\overline{\rho}(\tau)\overline{\rho}(\phi)^{-1} = \overline{\rho}(\tau)^q,$$

we deduce that $\overline{\Phi}\,\overline{N}\,\overline{\Phi}^{-1} = q\overline{N}$. Therefore we study the functor $M_{\overline{N}}$ on $\widehat{\mathcal{C}}_{\mathcal{O}}$ defined by

$$M_{\overline{N}}(R) = \{(\Phi, N) : N \in \mathrm{Nil}_{\overline{N}}(R),\ \Phi \in \widehat{G}_{\overline{\Phi}}(R),\ \Phi N \Phi^{-1} = qN\} \subset \mathrm{Nil}_{\overline{N}}(R) \times \widehat{G}_{\overline{\Phi}}(R).$$

Any such lift (Φ, N) to a coefficient ring R determines a homomorphism $T_q \to G(R)$ lifting $\overline{\rho}$ via $\phi \mapsto \Phi$ and $\tau \mapsto \exp(N)$, and it is continuous because $\exp(\overline{N})$ is unipotent. We will analyze $M_{\overline{N}}$ through the composition

$$M_{\overline{N}} \to \mathrm{Nil}_{\overline{N}} \to \mathrm{Spf}\, \mathcal{O}.$$

First, observe that $M_{\overline{N}} \to \mathrm{Nil}_{\overline{N}}$ is relatively representable as "$\Phi N = qN\Phi$" is a formal closed condition on points Φ of $(\widehat{G_{\overline{\Phi}}})_R$ for each $N \in \mathrm{Nil}_{\overline{N}}(R)$.

From Lemma 4.3.1.4, we know that $\mathrm{Nil}_{\overline{N}}$ is formally smooth over $\mathcal{O}$, and the universal nilpotent is $gN_\sigma g^{-1}$ for some $g \in \widehat{G}(\mathrm{Nil}_{\overline{N}})$. To check formal smoothness of the map $M_{\overline{N}} \to \mathrm{Nil}_{\overline{N}}$, it therefore suffices to check the formal smoothness of the fiber of $M_{\overline{N}}$ over the $\mathcal{O}$-point N_σ of $\mathrm{Nil}_{\overline{N}}$.

We have written down $\Phi_\sigma \in G(\mathcal{O})$ satisfying $\Phi_\sigma N_\sigma \Phi_\sigma^{-1} = qN_\sigma$ in Remark 4.2.2.9. Observe that $\overline{\Phi}\, \overline{\Phi}_\sigma^{-1} \in Z_G(N_\sigma)(k)$. By smoothness, we may lift $\overline{\Phi}\, \overline{\Phi}_\sigma^{-1}$ to an element $s \in Z_G(N_\sigma)(\mathcal{O})$. Then $s\Phi_\sigma$ reduces to $\overline{\Phi}$ and satisfies $(s\Phi_\sigma)N_\sigma(s\Phi_\sigma)^{-1} = qN_\sigma$, so the fiber of $M_{\overline{N}}$ over N_σ has an $\mathcal{O}$-point. The relative dimension of the formally smooth $\mathrm{Nil}_{\overline{N}}$ is $\dim G_k - \dim Z_{G_k}(\overline{N})$ by Lemma 4.3.1.4, and $M_{\overline{N}} \to \mathrm{Nil}_{\overline{N}}$ is a $\widehat{Z_G(N_\sigma)}$-torsor since it has an $\mathcal{O}$-point over N_σ. As $Z_G(N_\sigma)$ is smooth it follows that $M_{\overline{N}}$ is formally smooth over $\mathrm{Spf}\, \mathcal{O}$ of relative dimension $\dim G_k$. □

Example 4.4.2.4. This recovers [CHT08, Lemma 2.4.19] in the case $G = \mathrm{GL}_n$.

Let S be the (torus) quotient of G by its derived group G', and $\mu : G \to S$ the quotient map. For use elsewhere, we now study a variant where we fix a lift $\nu : T_q \to S(\mathcal{O})$ of $\mu \circ \overline{\rho} : T_q \to S(k)$:

Corollary 4.4.2.5. *The deformation condition of minimally ramified lifts $\rho : T_q \to G(R)$ satisfying $\mu \circ \rho = \nu$ is a liftable deformation condition. The framed deformation ring $R_{\overline{\rho}}^{\mathrm{m.r.},\nu,\square}$ is of relative dimension* $\dim G_k - 1$.

Proof. The quotient torus $S = G/G'$ is split of rank 1, so the subscheme $R_{\overline{\rho}}^{\mathrm{m.r.},\nu,\square} \subset R_{\overline{\rho}}^{\mathrm{m.r.}\,\square}$ is the vanishing of locus of a single function. As $R_{\overline{\rho}}^{\mathrm{m.r.}\,\square}$ is formally smooth over $\mathcal{O}$ with relative dimension $\dim G_k$, it suffices to check that the "reduced" tangent space of $R_{\overline{\rho}}^{\mathrm{m.r.},\nu,\square}$ over k is a proper subspace of the "reduced" tangent space of $R_{\overline{\rho}}^{\mathrm{m.r.}\,\square}$.

Let Z be the maximal central torus of G. On the level of Lie algebras, we know that $\mathrm{Lie}\, G$ splits over $\mathcal{O}$ as a direct sum of $\mathrm{Lie}\, G'$ and $\mathrm{Lie}\, S \simeq \mathrm{Lie}\, Z$ as p is very good for G. We can modify a lift ρ_0 over $R = k[\epsilon]/(\epsilon^2)$ by multiplying against an unramified non-trivial character $T_q \to Z(R)$ with trivial reduction, changing $\mu \circ \rho_0$. Thus the "reduced" tangent space of $R_{\overline{\rho}}^{\mathrm{m.r.},\nu,\square}$ is a proper subspace of that of $R_{\overline{\rho}}^{\mathrm{m.r.}\,\square}$ □

4.4.3 Deformation Conditions Based on Parabolic Subgroups

The use of nilpotent orbits is not the only approach to defining a minimally ramified deformation condition. As discussed in §1.2.3, the method used to prove [CHT08, Lemma 2.4.19] suggests a generalization from GL_n to other groups G based on deformations lying in certain parabolic subgroups of G. This deformation condition is not smooth for algebraic groups beyond GL_n, so it does not work in Ramakrishna's method. In this section we give a conceptual explanation for this phenomenon.

Let $P \subset G$ be a parabolic $\mathcal{O}$-subgroup. The Richardson orbit for P_k intersects $(\mathrm{Lie}\, P)_k$ in a dense open set which is a single geometric orbit under P_k. Suppose that $\overline{\rho}(\tau)$ is the exponential of a k-point $\overline{N}$ in the Richardson orbit, and consider deformations $\rho : T_q \to G(\mathcal{O})$ of $\overline{\rho}$ ramified with respect to P in the sense that $\rho(\tau) \in P$ (compare with Definition 1.2.3.1). This requires specifying a lift of $\overline{N}$ that lies in $\mathrm{Lie}\, P$. One could hope that such lifts would automatically be $G(\mathcal{O})$-conjugate to the fixed lift N_σ defined in Proposition 4.1.2.8, reminiscent of the definition we gave for $\mathrm{Nil}_{\overline{N}}$, a situation in which the associated (framed) deformation ring is smooth.

We now show that often smoothness fails if $\overline{N}$ does *not* lie in the Richardson orbit of P_k. Lifts of $\overline{N}$ can "change nilpotent type" yet still lie in a parabolic lifting P_k, such as the example of the standard Borel subgroup in GL_3 with

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \quad \text{lifting} \quad \overline{N} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In particular, we easily obtain non-pure nilpotents. This is very bad: the nilpotent orbits over a field are smooth but the nilpotent cone is not smooth, so the deformation problem of deforming with respect to P should not be smooth because "it sees multiple orbits". Furthermore, by working with lifts to $\mathcal{O}$ even if we could lift $\overline{\rho}(\tau)$ appropriately, there would still be problems lifting the image of ϕ because the centralizer of a non-pure nilpotent is not smooth over $\mathcal{O}$ (the special and generic fiber typically have different dimensions). So it is crucial to choose a parabolic such that $\overline{N}$ lies in the Richardson orbit of P_k.

For GL_n, *all* nilpotent orbits are Richardson orbits. This is not true in general. In particular, we should not expect the deformation condition of being ramified with respect to a parabolic to be liftable. Example 1.2.3.2 illustrates this phenomenon for GSp_4, which we now revisit in a more conceptual manner.

Example 4.4.3.1. Take $G = \mathrm{GSp}_4$. Parabolic subgroups correspond to isotropic flags. Up to conjugacy, these subgroups are G (the trivial parabolic) and stabilizers of the flags

$$0 \subset \mathrm{Span}(v_1) \subset \mathrm{Span}(v_1, v_2) \subset \mathrm{Span}(v_1, v_2, v_3) \subset k^4,$$
$$0 \subset \mathrm{Span}(v_1) \subset \mathrm{Span}(v_1, v_2, v_3) \subset k^4, \quad 0 \subset \mathrm{Span}(v_1, v_2) \subset k^4$$

where $\{v_1, v_2, v_2, v_4\}$ is the standard basis of k^4. Their Richardson orbits correspond to the respective nilpotent orbits indexed by the partitions $1+1+1+1$, 4, 4, and $2+2$. In particular, the same Richardson orbit is associated to two of these. There is one more partition of 4 with odd numbers appearing an even number of times: $2+1+1$. This is a nilpotent orbit that is not a Richardson orbit; for the representation in Example 1.2.3.2, $\log(\overline{\rho}(\tau))$ is in this nilpotent orbit.

4.5 Minimally Ramified Deformations of Symplectic and Orthogonal Groups

We continue the notation of the previous section. We have defined the minimally ramified deformation condition for representations factoring through the quotient $T_q = \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$ of the tame Galois group Γ_L^{t} at a place away from p. In this section, we will adapt the matrix-theoretic methods in [CHT08, §2.4.4], making use of more conceptual module-theoretic arguments, to define the minimally ramified deformation condition for any representation when $G = \mathrm{GSp}_m$ or $G = \mathrm{GO}_m$. (Minor variants of this method work for Sp_m and SO_m, and the original method of [CHT08, §2.4.4] works for GL_m.) We naturally embed G into $\mathrm{GL}(M)$ for a free $\mathcal{O}$-module M of rank m, and let V denote the reduction of M, a vector space over the residue field k.

We consider a representation $\overline{\rho} : \Gamma_L \to G(k) \subset \mathrm{GL}(V)(k)$ which may be wildly ramified (with L an ℓ-adic field for $\ell \neq p$). We will define a deformation condition for $\overline{\rho}$ in terms of the minimally ramified deformation condition for certain associated tamely ramified representations, after possibly extending $\mathcal{O}$. In §4.5.1, we analyze $\overline{\rho}$ as being built out of two pieces of data: representations of a closed normal subgroup Λ_L of Γ_L whose pro-order is prime to p, and tamely ramified representations of Γ_L/Λ_L. The representation theory of Λ_L is manageable since its pro-order is prime to p, and representations of Γ_L/Λ_L can be understood using the results of the previous section.

4.5.1 Decomposing Representations

We begin with a few preliminaries about representations over rings. Let Λ' be a profinite group and R be an Artinian coefficient ring with residue field k. If Λ' has pro-order prime to p, the representation theory over k is nice: every finite-dimensional continuous representation is a direct sum of irreducibles, and every such representation is projective over $k[\Lambda]$ for any finite discrete quotient Λ of Λ' through which the representation factors. We are also interested in corresponding statements over an Artinian coefficient ring R.

Fact 4.5.1.1. *Suppose the pro-order of Λ' is prime to p. Let P and P' be $R[\Lambda']$-modules that are finitely generated over R with continuous action of Λ', and F be a $k[\Lambda']$-module that is finite dimensional over k with continuous action of Λ'. Let Λ be a finite discrete quotient of Λ' through which the Λ'-actions on P, P', and F factor.*

1. *If P is free as an R-module, it is projective as a $R[\Lambda]$-module.*

2. *If P and P' are projective over $R[\Lambda]$, they are isomorphic if and only if $\overline{P}$ and $\overline{P'}$ are isomorphic.*

3. *There exists a projective $R[\Lambda]$-module (unique up to isomorphism) whose reduction is F.*

These statements are special cases of results in [Ser77, §14.4]. We now record two lemmas which do not need the assumption that the pro-order of Λ' is prime to p.

Lemma 4.5.1.2. *Let P and P' be $R[\Lambda']$-modules, finitely generated over R with continuous action of Λ' factoring through a finite discrete quotient Λ of Λ'. Assume P and P' are $R[\Lambda]$-projective. The natural map gives an isomorphism*

$$\mathrm{Hom}_{\Lambda'}(P, P') \otimes_R k \to \mathrm{Hom}_{\Lambda'}(\overline{P}, \overline{P'}).$$

Proof. We may replace $\mathrm{Hom}_{\Lambda'}$ with Hom_{Λ}. Note that $\mathfrak{m}P' = \mathfrak{m} \otimes_R P'$, so $\mathrm{Hom}_{\Lambda}(P, \mathfrak{m}P') = \mathrm{Hom}_{\Lambda}(P, P') \otimes_R \mathfrak{m}$ as P and P' are $R[\Lambda]$-projective. Then apply $\mathrm{Hom}_{\Lambda}(P, -)$ to the exact sequence $0 \to \mathfrak{m}P' \to P' \to P'/\mathfrak{m}P' \to 0$. □

Lemma 4.5.1.3. *Let Λ be a finite group and let M and M' be finite $R[\Lambda]$-modules whose reductions $\overline{M}$ and $\overline{M}'$ are non-isomorphic irreducible $k[\Lambda]$-modules. Then $\mathrm{Hom}_{R[\Lambda]}(M, M') = 0$.*

Proof. Filter M' by the composition series $\{\mathfrak{m}^i M'\}$, and consider the surjection

$$\mathfrak{m}^i/\mathfrak{m}^{i+1} \otimes \overline{M}' \twoheadrightarrow \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M'.$$

The action of Λ on $\mathfrak{m}^i/\mathfrak{m}^{i+1} \otimes M'$ is solely on the irreducible $\overline{M}'$, so as a $k[\Lambda]$-module $\mathfrak{m}^i M'/\mathfrak{m}^{i+1} M'$ is isomorphic to a direct sum of copies of $\overline{M}'$. Thus

$$\mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M') = \mathrm{Hom}_{k[\Lambda]}(\overline{M}, \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M') = 0$$

as $\overline{M}$ and $\overline{M}'$ are non-isomorphic $k[\Lambda]$-modules.

By descending induction on i, we shall show that

$$\mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^i M') = 0.$$

For large i, $\mathfrak{m}^i M' = 0$. Consider the exact sequence

$$0 \to \mathfrak{m}^{i+1} M' \to \mathfrak{m}^i M' \to \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M' \to 0.$$

Applying $\mathrm{Hom}_{R[\Lambda]}(M, -)$, we obtain a left exact sequence

$$0 \to \mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^{i+1} M') \to \mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^i M') \to \mathrm{Hom}_{R[\Lambda]}(M, \mathfrak{m}^i M'/\mathfrak{m}^{i+1} M')$$

The left term is 0 by induction, and the right term is 0 by the above calculation. This completes the induction. □

Given $\overline{\rho} : \Gamma_L \to G(k) \subset \mathrm{GL}(V)(k)$ and a lift $\rho : \Gamma_L \to G(R) \subset \mathrm{GL}(M)(R)$ for some $R \in \mathcal{C}_{\mathcal{O}}$, we now turn to decomposing the $R[\Gamma_L]$-module M. Let $I_L \subset \Gamma_L$ be the inertia group, and pick a surjection $I_L \to \mathbf{Z}_p$. Define Λ_L to be the kernel of this surjection (normal in Γ_L). This is a pro-finite group with pro-order prime to p, and is independent of the choice of surjection. Define the quotient

$$T_L := \Gamma_L/\Lambda_L,$$

which is a quotient of the tamely ramified Galois group Γ_L^{t} and of the form $T_q = \widehat{\mathbf{Z}} \ltimes \mathbf{Z}_p$ as in §4.4. We wish to compatibly decompose V and M as Λ_L-modules and then understand the action of Γ_L on the decomposition.

We first make a finite extension of k (and hence of $\mathcal{O}$) so that all of the (finitely many) irreducible representations of Λ_L over k occuring in V are absolutely irreducible over k.

Because Λ_L has order prime to p, $\mathrm{Res}^{\Gamma_L}_{\Lambda_L}(V)$ is completely reducible and we can write

$$\mathrm{Res}^{\Gamma_L}_{\Lambda_L}(V) = \bigoplus_{\tau} V_\tau$$

where τ runs through the set Σ of isomorphism classes of irreducible representations of Λ_L over k occuring in V, and each V_τ is the τ-isotypic component. We will obtain an analogous decomposition for M.

Let Γ be a finite discrete quotient of Γ_L through which ρ factors, and let Λ be the image of Λ_L in Γ. Using Fact 4.5.1.1(3) we can lift τ to a projective $R[\Lambda]$-module $\widetilde{\tau}$ unique up to isomorphism. We will eventually want this lift to have additional properties (see §4.5.2), but this is not yet necessary. We set $W_\tau := \mathrm{Hom}_{\Lambda_L}(\widetilde{\tau}, M)$ and consider the natural morphism

$$\bigoplus_{\tau} \widetilde{\tau} \otimes_R W_\tau \to M.$$

Note that M is $R[\Lambda]$-projective by Fact 4.5.1.1(1).

Lemma 4.5.1.4. *This map is an isomorphism of $R[\Lambda_L]$-modules.*

Proof. It suffices to check it is an isomorphism of $R[\Lambda]$-modules. When $R = k$, we know $\mathrm{End}_\Lambda(\tau) = k$ as we extended k so that all of the irreducible representations of Λ over k occurring inside V are absolutely irreducible. Splitting up V as a direct sum of irreducibles, we obtain the desired isomorphism.

In the general case, the map is an isomorphism after reducing modulo $\mathfrak{m}$ (use Lemma 4.5.1.2). Thus by Nakayama's lemma it is surjective. Since M is R-projective, the formation of the kernel commutes with reduction modulo $\mathfrak{m}$. Thus, again using Nakayama's lemma the kernel is zero. ☐

We define M_τ to be the image of $\widetilde{\tau} \otimes_R \mathrm{Hom}_{\Lambda_L}(\widetilde{\tau}, M)$ in M. It is the largest $R[\Lambda_L]$-direct summand whose reduction is a direct sum of copies of τ.

We next seek to understand the action of Γ_L on this canonical decomposition of M. For $g \in \Gamma_L$, consider the $R[\Lambda_L]$-module gM_τ: it is a direct summand of M over R whose reduction is a direct sum of copies of the representation τ^g defined by $\tau^g(h) = \tau(g^{-1}hg)$ for $h \in \Lambda_L$. Thus we see that $gM_\tau = M_{\tau^g}$ inside M, and Γ_L permutes the M_τ's. The orbits corresponds to sets of conjugate representations.

Consider the stabilizer of V_τ:

$$\Gamma_{L,\tau} = \{g \in \Gamma_L : gV_\tau = V_\tau \text{ inside } V\} = \{g \in \Gamma_L : \tau^g \simeq \tau\} \subset \Gamma_L$$

with corresponding image

$$\Gamma_\tau = \{g \in \Gamma : gV_\tau = V_\tau \text{ inside } V\} = \{g \in \Gamma : \tau^g \simeq \tau\} \subset \Gamma.$$

Then the k-span of the members of the Γ_L-orbit of V_τ is exactly the representation $\mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} V_\tau = \mathrm{Ind}_{\Gamma_\tau}^{\Gamma} V_\tau$. Letting $[\tau]$ denote the set of $R[\Lambda_L]$-isomorphism classes of Λ-representations Γ_L-conjugate to τ, by taking into account the action of Γ_τ the isomorphism in Lemma 4.5.1.4 becomes an isomorphism of $R[\Gamma_L]$-modules

$$M = \bigoplus_{[\tau]} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau \tag{4.5.1.1}$$

using one representative τ per Γ_L-conjugacy class $[\tau]$.

For orthogonal or symplectic representations, this decomposition interacts with duality. Denote the similitude character by μ, and let $\overline{\nu} := \mu \circ \overline{\rho} : \Gamma_L \to k^\times$. Let N be a free $\mathcal{O}$-module of rank 1 on which Γ_L acts by a specified continuous $\mathcal{O}^\times$-valued lift ν of $\overline{\nu}$, and let $\overline{N}$ be its reduction modulo $\mathfrak{m}$. For an $\mathcal{O}$-module M, define $M^\vee = \mathrm{Hom}_{\mathcal{O}}(M, N)$ with the evident Γ_L-action. The perfect pairing gives an isomorphism of $R[\Gamma_L]$-modules $M \simeq M^\vee$. In particular,

$$M_\tau^\vee \simeq M_{\tau^*}$$

for some irreducible representation τ^* of Λ_L occuring in V. Note that $\tau^* \simeq \tau^\vee$ as $k[\Lambda_L]$-modules. There are three cases:

- **Case 1:** τ is not conjugate to τ^*;
- **Case 2:** τ is isomorphic to τ^*;
- **Case 3:** τ is conjugate to τ^* but not isomorphic.

In the second case, we claim that the isomorphism of $k[\Lambda_L]$-modules $\imath : \tau \simeq \tau^\vee$ gives a sign-symmetric perfect pairing on τ. Note that $W_\tau = \mathrm{Hom}_\Lambda(\tau, V) \simeq \mathrm{Hom}_\Lambda(\tau^\vee, V^\vee) \simeq W_\tau^\vee$ as $V_\tau \simeq V_\tau^\vee$. This isomorphism φ_τ defines a pairing $\langle , \rangle_{W_\tau}$ on W_τ via

$$\langle w_1, w_2 \rangle_{W_\tau} := \varphi_\tau(w_1)(w_2).$$

Let $\psi : V \to V^\vee$ be the isomorphism given by $m \mapsto \langle m, - \rangle_V$, and define $\langle v_1, v_2 \rangle_\tau := \imath(v_1)(v_2)$ for $v_1, v_2 \in \tau$. We have a commutative diagram

$$\begin{array}{ccccc} \tau \otimes W_\tau & \xrightarrow{\mathrm{id} \otimes \varphi_\tau} & \tau \otimes W_\tau^\vee & \xrightarrow{\imath \otimes \mathrm{id}} & \tau^\vee \otimes W_\tau^\vee \\ \downarrow & & & & \downarrow \\ V_\tau & & \xrightarrow{\quad\quad\psi\quad\quad} & & V_\tau^\vee \end{array}$$

The commutativity says that for elementary tensors $m_i = v_i \otimes w_i \in V_\tau = \tau \otimes W_\tau$ we have

$$\langle m_1, m_2 \rangle_M = \psi(m_1)(m_2) = (\imath(v_1) \otimes \varphi_\tau(w_1))\,(v_2 \otimes w_2) = \imath(v_1)(v_2) \cdot \varphi_\tau(w_1)(w_2) = \langle v_1, v_2 \rangle_\tau \langle w_1, w_2 \rangle_{W_\tau}. \tag{4.5.1.2}$$

Lemma 4.5.1.5. *The pairing $\langle , \rangle_\tau$ is a sign-symmetric.*

Proof. Suppose there exists $v \in \tau$ such that $\imath(v)(v) \neq 0$. For $w_1, w_2 \in W_\tau$, (4.5.1.2) gives

$$\imath(v)(v)\varphi_\tau(w_1)(w_2) = \langle v \otimes w_1, v \otimes w_2 \rangle_V = \epsilon \langle v \otimes w_2, v \otimes w_1 \rangle_V = \epsilon \imath(v)(v)\varphi_\tau(w_2)(w_1).$$

Canceling $\imath(v)(v)$, we conclude that $\langle w_1, w_2 \rangle_{W_\tau} = \epsilon \langle w_2, w_1 \rangle_{W_\tau}$. Using (4.5.1.2), we conclude that

$$\epsilon \imath(v_2)(v_1) \cdot \varphi_\tau(w_2)(w_1) = \epsilon \langle m_2, m_1 \rangle_V = \langle m_1, m_2 \rangle_V = \imath(v_1)(v_2) \cdot \varphi_\tau(w_1)(w_2) = \epsilon \imath(v_1)(v_2) \cdot \varphi_\tau(w_2)(w_1).$$

Choosing w_1 and w_2 with $\langle w_2, w_1 \rangle_{W_\tau} \neq 0$ (possible as $\langle , \rangle_V$ is perfect), we conclude that $\langle v_1, v_2 \rangle_{\widetilde{\tau}} = \langle v_2, v_1 \rangle_{\widetilde{\tau}}$.

Otherwise $\imath(v)(v) = 0$ for all $v \in \tau$, in which case $\langle , \rangle_{\widetilde{\tau}}$ is alternating. ☐

In §4.5.2 we will see that the action of Λ_L on the module underlying $\widetilde{\tau}$ can be extended to an action of $\Gamma_{L,\tau}$ factoring through Γ_τ. Therefore, $W_\tau = \mathrm{Hom}_{\Lambda_L}(\widetilde{\tau}, M)$ is naturally a representation of $T_{L,\tau} := \Gamma_{L,\tau}/\Lambda_L$, and of $T_\tau := \Gamma_\tau/\Lambda$ (a finite quotient of $T_{L,\tau}$). In §4.5.3, we will use the minimally ramified deformation condition of §4.4 to specify which deformations W_τ are allowed. Together with the decomposition (4.5.1.1)

$$\bigoplus_{[\tau]} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\widetilde{\tau} \otimes W_\tau) \to M$$

this defines a deformation condition for $\overline{\rho}$. Some care is needed to ensure compatibility with the pairing on M, which will require breaking into cases in the next sections.

4.5.2 Extension of Representations

We continue the notation of the previous section, where τ is an absolutely irreducible representation of Λ_L over k. We need to lift this to a representation over $\mathcal{O}$ and extend it to a representation of $\Gamma_{L,\tau}$. We will have to do something extra for the representation to be compatible with a pairing, depending on how τ and τ^* are related.

In **Case 1**, we ignore the pairing. The results of [CHT08, §2.4.4] let us pick a $\mathcal{O}[\Gamma_{L,\tau}]$-module $\widetilde{\tau}$ that is a free $\mathcal{O}$-module and reduces to τ. In this case, $\widetilde{\tau}^\vee$ is a free $\mathcal{O}$-module reducing to τ^*.

In **Case 2**, from Lemma 4.5.1.5 it follows that τ is a symplectic or orthogonal representation. We will adapt the GL_n-technique of [CHT08] to produce a symplectic or orthogonal extension $\widetilde{\tau}$. Letting $n = \dim \tau$, the representation τ gives a homomorphism $\tau : \Lambda_L \to G(k)$ where G is GSp_n or GO_n.

First, we claim that there is a continuous lift $\widetilde{\tau} : \Lambda_L \to G(W(k))$: without the pairing, this would be Fact 4.5.1.1(3). To also take into account the pairing, consider deformation theory for the residual representation τ. This is a smooth deformation condition as $H^2(\Lambda_L, \mathrm{ad}\,\tau) = 0$: Λ_L has pro-order prime to p and $\operatorname{ad}\tau$ has order a power of p. Therefore the desired lift exists. It is unique (up to conjugations which preserve τ) because the tangent space is zero dimensional as $H^1(\Lambda_L, \operatorname{ad}\tau) = 0$. By considering representations of the group $\Lambda_L/\ker(\tau)$, we may and do assume arrange that $\ker(\widetilde{\tau}) = \ker(\tau)$ as subgroups of Λ_L.

Remark 4.5.2.1. For $g \in \Gamma_{L,\tau}$, the isomorphism of $k[\Lambda_L]$-modules $\tau^g \simeq \tau$ lifts to an isomorphism $\widetilde{\tau}^g \simeq \widetilde{\tau}$ of $\mathcal{O}[\Lambda_L]$-modules by uniqueness. Thus $\Gamma_{L,\tau} = \{g \in \Gamma_L : \widetilde{\tau}^g \simeq \widetilde{\tau}\}$.

We will now show how to continuously extend $\widetilde{\tau}$ to $\Gamma_{L,\tau}$. The first step in constructing the extension is to understand the structure of $\Gamma_{L,\tau}$ and $I_L \cap \Gamma_{L,\tau}$, where I_L is the inertia group.

Recall that $T_L = \Gamma_L/\Lambda_L$ is the semi-direct product of $\widehat{\mathbf{Z}}$ and $\mathbf{Z}_p$, where $\widehat{\mathbf{Z}}$ is generated by a lift of Frobenius ϕ and $\mathbf{Z}_p$ is generated by an element σ, with $\phi\sigma\phi^{-1} = \sigma^q$ where $q = \ell^a$ is the size of the residue field of L.

Lemma 4.5.2.2. *The exact sequence*

$$1 \to \Lambda_L \to \Gamma_L \to T_L \to 1$$

is topologically split, so Γ_L is a semi-direct product.

Proof. Let S be a Sylow pro-p subgroup of I_L, which must be isomorphic to $\mathbf{Z}_p$. Let ϕ be a lift of Frobenius to Γ_L. Then $\phi S\phi^{-1}$ is another Sylow pro-p subgroup of I_L, and hence is conjugate to S by an element of I_L. By choosing the lift ϕ, we may thereby assume that ϕ normalizes S. But then it is clear that S and ϕ together topologically generate T_L, giving the desired splitting. □

For $T_{L,\tau} := \Gamma_{L,\tau}/\Gamma_L$, this gives a topological splitting of

$$1 \to \Lambda_L \to \Gamma_{L,\tau} \to T_{L,\tau} \to 1.$$

As $\Gamma_{L,\tau}$ is an open subgroup of Γ_L, we observe that $T_{L,\tau}$ is an open subgroup of T_L. Note that $T_{L,\tau}$ is an open normal subgroup of T_L topologically generated by some powers of ϕ and σ which will be denoted by ϕ_τ and σ_τ (since any open subgroup of a semidirect product $C \ltimes C'$ for pro-cyclic C and C' is of the form $C_0 \ltimes C_0'$ for open subgroups $C_0 \subset C$ and $C_0' \subset C'$). In particular, $T_{L,\tau}$ is itself isomorphic to $T_{q'}$ for some q'. The element σ_τ and Λ_L together topologically generate $\Gamma_{L,\tau} \cap I_L$.

Before extending $\widetilde{\tau}$, we need several technical lemmas.

Lemma 4.5.2.3. *The centralizer of the image of $\widetilde{\tau}$ is $\mathcal{O}$.*

Proof. As τ is absolutely irreducible, $\operatorname{End}_{\Lambda_L}(\tau) = k$. By Lemma 4.5.1.2, we see that the reduction of $\operatorname{End}_{\Lambda_L}(\widetilde{\tau})$ modulo the maximal ideal of $\mathcal{O}$ is k, so the map $\mathcal{O} \hookrightarrow \operatorname{End}_{\Lambda_L}(\widetilde{\tau})$ is surjective by Nakayama's lemma. □

Lemma 4.5.2.4. *The dimension of τ is not divisible by p.*

Proof. As τ is continuous and Λ_L has pro-order prime to p, the representation τ factors through a finite discrete quotient Λ of Λ_L whose order is prime to p. Such a representation is the reduction of a projective $\mathcal{O}[\Lambda]$-module by Fact 4.5.1.1(3). Inverting p, we obtain a representation of Λ in characteristic zero that is absolutely irreducible since the "reduction" τ is absolutely irreducible over k. By [Ser77, §6.5 Corollary 2], the dimension of this representation (equal to the dimension of τ) divides the order of Λ. □

We will now extend $\widetilde{\tau}$ from $\Lambda_L \subset I_L$ to $\Gamma_{L,\tau}$ by defining it on the topological generators σ_τ and ϕ_τ. We say that such an extension has *tame determinant* if $\det(\widetilde{\tau}(\sigma_\tau))$ has finite order which is prime to p.

Lemma 4.5.2.5. *There is a unique continuous extension $\widetilde{\tau} : \Gamma_{L,\tau} \cap I_L \to G(\mathcal{O})$ with tame determinant.*

Proof. A continuous extension of $\widetilde{\tau}$ to $\Gamma_{L,\tau} \cap I_L$ is determined by its value on σ_τ. As $\sigma_\tau \in \Gamma_{L,\tau}$, in light of Remark 4.5.2.1 there is an $A \in G(\mathcal{O})$ such that for $g \in \Lambda_L$ we have

$$\widetilde{\tau}(\sigma_\tau g \sigma_\tau^{-1}) = A\widetilde{\tau}(g)A^{-1}.$$

We would like to send σ_τ to the element A. However, this might not produce a continuous extension, and even if it does it might not have tame determinant unless we pick A correctly. As σ_τ is a topological generator for a group isomorphic to $\mathbf{Z}_p$, the existence of a continuous extension satisfying $\sigma_\tau \mapsto A$ is equivalent to some p-power of A having trivial reduction. We wish to show that there is a unique choice of such A that also makes the extension have tame determinant.

We will first show that some power A^{p^b} lies in the centralizer of the image $\widetilde{\tau}(\Lambda_L)$. Consider the conjugation action of $\langle\sigma_\tau\rangle$ on Λ_L. As $\ker\widetilde{\tau} = \ker\tau$ is a normal subgroup of $\Gamma_{L,\tau}$ (if $g \in \Gamma_{L,\tau}$ and $\tau(h) = 1$, then $\tau^g(h)$ is conjugate to $\tau(h) = 1$ by Remark 4.5.2.1) we get an action of $\langle\sigma_\tau\rangle$ on $\Lambda_L/\ker\tau \simeq \tau(\Lambda_L)$. The action is continuous, so there is a power p^b such that for all $g \in \Lambda_L$ we have

$$\tau(\sigma_\tau^{p^b} g \sigma_\tau^{-p^b}) = \tau(g).$$

As $\ker\widetilde{\tau} = \ker\tau$, we see that

$$A^{p^b}\widetilde{\tau}(g)A^{-p^b} = \widetilde{\tau}(\sigma_\tau^{p^b} g \sigma_\tau^{-p^b}) = \widetilde{\tau}(g).$$

Therefore A^{p^b} lies in the centralizer of $\widetilde{\tau}(\Lambda_L)$.

By Lemma 4.5.2.3, this centralizer is $\mathcal{O}$. We claim that by multiplying A by some unit in $\mathcal{O}$, we can arrange for the continuous extension $\widetilde{\tau}$ to exist and have tame determinant. We will use the fact that an element of $\mathcal{O}^\times$ is the product of a 1-unit and a Teichmuller lift of an element of $k^\times$. As $A^{p^b} \in \mathcal{O}^\times$ and the pth power map is an automorphism of $k^\times$, we may multiply A by a unit scalar so that A^{p^b} reduces to the identity matrix. By Lemma 4.5.2.4, the dimension n of τ is prime to p so we may multiply A by a 1-unit so that $\det(A)$ has finite order prime to p. Sending σ_τ to this particular A gives a continuous extension with tame determinant.

Let's show this extension is unique. Any extension must send σ_τ to an element of the form wA for $w \in \mathcal{O}^\times$ (the centralizer of the image $\widetilde{\tau}(\Lambda_L)$). By continuity, there is a power p^b such that $(wA)^{p^b}$ reduces to the identity. This means that w^{p^b} reduces to the identity, and hence that w reduces to the identity. On the other hand, $\det(wA)\det(A)^{-1} = w^n$. The left side has finite order that is relatively prime to p, so w^n does too. This forces $w^n = 1$ since its reduction is 1. But as n is prime to p (Lemma 4.5.2.4), the only nth roots of unity in $\mathcal{O}^\times$ are Teichmuller lifts. Therefore $w = 1$. ☐

Lemma 4.5.2.6. *There is a continuous extension* $\widetilde{\tau} : \Gamma_{L,\tau} \to G(\mathcal{O})$.

Proof. We extend $\widetilde{\tau}$ in Lemma 4.5.2.5 continuously to $\Gamma_{L,\tau}$ by defining it on ϕ_τ. As $\phi_\tau \in \Gamma_{L,\tau}$, there is an element $A \in G(\mathcal{O})$ conjugating $\widetilde{\tau} : \Lambda_L \to G(\mathcal{O})$ to $\widetilde{\tau}^{\phi_\tau} : \Lambda_L \to G(\mathcal{O})$. Each has a unique extension to a continuous morphism from $I_L \cap \Gamma_{L,\tau}$ to $G(\mathcal{O})$ with tame determinant. Therefore for $g \in I_L \cap \Gamma_{L,\tau}$ we have

$$\widetilde{\tau}(\phi_\tau g \phi_\tau^{-1}) = A\widetilde{\tau}(g)A^{-1}$$

since the right side has the same (tame) determinant as $\widetilde{\tau}$ on $I_L \cap T_\tau$. We can continuously extend $\widetilde{\tau} : I_L \cap \Gamma_{L,\tau} \to G(\mathcal{O})$ by sending ϕ_τ to A since A has reduction with finite order. ☐

This gives the desired lift and extension of τ in the case that $\tau \simeq \tau^*$.

In **Case 3**, τ is conjugate to τ^* but not isomorphic. The argument follows the same structure as the previous case, but we make a few modifications to treat $\tau \oplus \tau^*$ together. In particular, we can pick a copy of the $k[\Lambda_L]$-module τ inside V_τ and a copy of $\tau^* \simeq \tau^\vee$ inside V such that the pairing restricted to $\tau \oplus \tau^*$ is perfect.

Define $\Gamma_{L,\tau\oplus\tau^*} = \{g \in \Gamma_L : g(\tau \oplus \tau^*) = \tau \oplus \tau^*\}$. It contains $\Gamma_{L,\tau}$ with index 2, as any automorphism either preserves τ and τ^* or swaps them. Arguing as in the paragraph after Lemma 4.5.2.2, we obtain a split exact sequence

$$0 \to \Lambda_L \to \Gamma_{L,\tau\oplus\tau^*} \to T_{L,\tau\oplus\tau^*} \to 1$$

where $T_{L,\tau\oplus\tau^*}$ is an open normal subgroup of T_L topologically generated by some powers of ϕ and σ which we denote by $\phi_{\tau\oplus\tau^*}$ and $\sigma_{\tau\oplus\tau^*}$. We may arrange that either

- **Case 3a:** $\phi^2_{\tau\oplus\tau^*} = \phi_\tau$ and $\sigma_{\tau\oplus\tau^*} = \sigma_\tau$ or
- **Case 3b:** $\phi_{\tau\oplus\tau^*} = \phi_\tau$ and $\sigma^2_{\tau\oplus\tau^*} = \sigma_\tau$.

In **Case 3a**, we begin by lifting τ to $\mathcal{O}$ as a representation of Λ_L: as before, we do this using the fact that the pro-order of Λ_L is prime to p, and obtain a lift $\widetilde{\tau}$ unique up to isomorphism. We extend $\widetilde{\tau}$ to be a representation of $\Gamma_{L,\tau} \cap I_L$ by defining it on σ_τ using the GL_n-version of Lemma 4.5.2.5, [CHT08, Lemma 2.4.11]. There it is shown all such extensions are unique up to equivalence. In particular, $\widetilde{\tau}$ and $(\widetilde{\tau}^{\phi_{\tau\oplus\tau^*}})^\vee$ are isomorphic $\mathcal{O}[\Gamma_{L,\tau} \cap I_K]$-modules. We can use this to define a sign-symmetric perfect pairing on $\widetilde{\tau} \oplus \widetilde{\tau}^{\phi_{\tau\oplus\tau^*}}$ that is compatible with the action of $\Gamma_{L,\tau} \cap I_K$ and $\phi_{\tau\oplus\tau^*}$, hence of $\Gamma_{L,\tau\oplus\tau^*}$.

In **Case 3b**, as $\tau^\vee$ and $\tau^{\sigma_{\tau\oplus\tau^*}}$ are isomorphic $k[\Lambda_L]$-modules it follows that $\widetilde{\tau}^\vee$ and $\widetilde{\tau}^{\sigma_{\tau\oplus\tau^*}}$ are isomorphic $\mathcal{O}[\Lambda_L]$-modules. In particular, this isomorphism gives a natural way to define a sign-symmetric perfect pairing on $M = \widetilde{\tau} \oplus \widetilde{\tau}^{\sigma_{\tau\oplus\tau^*}}$ lifting the residual one. This pairing is compatible with the action of $\Gamma_{L,\tau\oplus\tau^*} \cap I_L$ (which is generated by Λ_L and $\sigma_{\tau\oplus\tau^*}$). Finally, we claim that M and M^{ϕ_τ} are isomorphic. As $\phi_\tau \in \Gamma_{L,\tau}$ preserves τ, the reductions of M and M^{ϕ_τ} are isomorphic by an isomorphism which identifies τ and τ^{ϕ_τ}. By uniqueness of the lift of τ as a $\mathcal{O}[\Lambda_L]$-module, we obtain an isomorphism of $\widetilde{\tau}$ and $\widetilde{\tau}^{\phi_\tau}$ and hence of M and M^{ϕ_τ} preserving the pairing. Then we proceed as in the proof of Lemma 4.5.2.6, defining an image of ϕ_τ using this isomorphism.

In conclusion, we have shown:

Lemma 4.5.2.7. *In case 3, there exists an $\mathcal{O}[\Gamma_{L,\tau\oplus\tau^*}]$-module $\widetilde{\tau \oplus \tau^*}$ with pairing lifting $\tau \oplus \tau^*$ together with its pairing.*

4.5.3 Lifts with Pairings

We continue the notation of §4.5.1, and analyze how the duality pairing interacts with the decomposition (4.5.1.1). Recall that we obtained an isomorphism $M \simeq M^\vee$ of $R[\Gamma_L]$-modules which gave isomorphisms $M_\tau \simeq M^\vee_{\tau^*}$ of $R[\Gamma_{L,\tau}]$-modules. The key point is that for any lift and extension τ' of τ to an $\mathcal{O}[\Gamma_{L,\tau}]$-module, the isomorphism of $R[\Lambda_L]$-modules

$$\tau' \otimes \mathrm{Hom}_{\Lambda_L}(\tau', M) \to M_\tau$$

is compatible with the $\Gamma_{L,\tau}$-action.

To do this, it is convenient to break into the cases introduced at the end of §4.5.1. For an irreducible $k[\Lambda]$-module τ occurring in V, note that $(\tau^g)^\vee = (\tau^\vee)^g$ for any $g \in \Gamma_L$, so if $\tau \simeq \tau^*$ then $\tau^g \simeq (\tau^g)^*$. We let

- Σ_n denote the set of Γ_L-conjugacy classes of such τ for which τ is not conjugate to τ^*;
- Σ_e denote the set of Γ_L-conjugacy classes of such τ for which $\tau \simeq \tau^*$;
- Σ_c denote the set of Γ_L-conjugacy classes of such τ for which τ^* is conjugate to τ but $\tau \not\simeq \tau^*$.

From (4.5.1.1), we obtain a decomposition

$$M = \bigoplus_{\tau\in\Sigma_n} \mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau}} (\tau' \otimes W_\tau) \oplus \bigoplus_{\tau\in\Sigma_e} \mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau}} (\tau' \otimes W_\tau) \oplus \bigoplus_{\tau\in\Sigma_c} \mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau}} (\tau' \otimes W_\tau) \tag{4.5.3.1}$$

where τ' is any lift and extension of τ to Γ_τ and $W_\tau = \mathrm{Hom}_\Lambda(\tau', M)$ is a representation of $T_{L,\tau}$. Note that W_τ is free as an R-module (since M and τ' are, with $\tau' \neq 0$ and R local), and hence that W_τ is tamely ramified of the type considered in §4.4.

We may rewrite this to make use of the special extensions constructed in §4.5.2. In particular, for $\tau \in \Sigma_c$ we rewrite

$$\mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau}} (\tau' \otimes W_\tau) = \mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau\oplus\tau^*}} \left(\widetilde{\tau \oplus \tau^*} \otimes W_{\tau\oplus\tau^*}\right)$$

where $W_{\tau\oplus\tau^*} := \mathrm{Hom}_{\Lambda'_L}(\widetilde{\tau \oplus \tau^*}, M)$. This uses the notation and results from Case 3 in §4.5.2, in particular the fact that $\tau \oplus \tau^*$ is an irreducible representation of the group Λ'_L generated by Λ_L and any $g \in \Gamma_L$ with $\tau^* \simeq \tau^g$. Note that $W_{\tau\oplus\tau^*}$ is a representation of $T_{L,\tau\oplus\tau^*}$, which is a subgroup of $T_L = \Gamma_L/\Lambda_L$, hence of the form T_q as considered in §4.4. Using the extensions $\widetilde{\tau}$ and $\widetilde{\tau \oplus \tau^*}$ from §4.5.2, we obtain a decomposition

$$M = \bigoplus_{\tau\in\Sigma_n} \mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau}} (\widetilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau\in\Sigma_e} \mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau}} (\widetilde{\tau} \otimes W_\tau) \oplus \bigoplus_{\tau\in\Sigma_c} \mathrm{Ind}^{\Gamma_L}_{\Gamma_{L,\tau\oplus\tau^*}} \left(\widetilde{\tau \oplus \tau^*} \otimes W_{\tau\oplus\tau^*}\right). \tag{4.5.3.2}$$

Now let M' be another $R[\Gamma_L]$-module that is finite free over R such that the irreducible representations of Λ_L occurring in $V' := M'/\mathfrak{m}M'$ are among the same τ's, so

$$M' = \bigoplus_{\tau\in\Sigma_n} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\widetilde{\tau}\otimes W'_\tau) \oplus \bigoplus_{\tau\in\Sigma_e} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} (\widetilde{\tau}\otimes W'_\tau) \oplus \bigoplus_{\tau\in\Sigma_c} \mathrm{Ind}_{\Gamma_{L,\tau\oplus\tau^*}}^{\Gamma_L} \left(\widetilde{\tau\oplus\tau^*}\otimes W'_{\tau\oplus\tau^*}\right).$$

Lemma 4.5.3.1. *The natural map*

$$\bigoplus_{\tau\in\Sigma_n} \mathrm{Hom}_{T_{L,\tau}}(W_\tau, W'_\tau) \oplus \bigoplus_{\tau\in\Sigma_e} \mathrm{Hom}_{T_{L,\tau}}(W_\tau, W'_\tau) \oplus \bigoplus_{\tau\in\Sigma_c} \mathrm{Hom}_{T_{L,\tau\oplus\tau^*}}(W_{\tau\oplus\tau^*}, W'_{\tau\oplus\tau^*}) \to \mathrm{Hom}_{\Gamma_L}(M, M')$$

is an isomorphism.

Proof. We may immediately pass to working with representations of the finite discrete groups Γ and Λ. Notice that

$$\mathrm{Hom}_\Gamma(\mathrm{Ind}_{\Gamma_\tau}^\Gamma(M_\tau), \mathrm{Ind}_{\Gamma_\tau}^\Gamma(M'_\tau)) \simeq \mathrm{Hom}_{\Gamma_\tau}(M_\tau, \mathrm{Ind}_{\Gamma_\tau}^\Gamma(M'_\tau)) \simeq \mathrm{Hom}_{\Gamma_\tau}(M_\tau, M'_\tau)$$

where the second isomorphism uses that $\mathrm{Hom}_{\Gamma_\tau}(M_\tau, M'_{\tau^g}) = 0$ by Lemma 4.5.1.3 when τ and τ^g are non-isomorphic. Furthermore, if τ_1 and τ_2 are not Γ-conjugate then

$$\mathrm{Hom}_\Gamma(\mathrm{Ind}_{\Gamma_{\tau_1}}^\Gamma(M_{\tau_1}), \mathrm{Ind}_{\Gamma_{\tau_2}}^\Gamma(M'_{\tau_2})) = 0$$

again using Lemma 4.5.1.3. Then using (4.5.1.1) we see that

$$\mathrm{Hom}_\Gamma(M, M') = \bigoplus_{[\tau_1],[\tau_2]} \mathrm{Hom}_\Gamma(\mathrm{Ind}_{\Gamma_{\tau_1}}^\Gamma(M_{\tau_1}), \mathrm{Ind}_{\Gamma_{\tau_2}}^\Gamma(M'_{\tau_2})) = \bigoplus_{[\tau]} \mathrm{Hom}_{\Gamma_\tau}(M_\tau, M'_\tau).$$

All the irreducible finite-dimensional representations of Λ occurring in V and V' are absolutely irreducible over k by design. For $\tau\in\Sigma_n\cup\Sigma_e$, consider the natural inclusion

$$\mathrm{Hom}_R(W_\tau, W'_\tau) \hookrightarrow \mathrm{Hom}_\Lambda(\widetilde{\tau}\otimes W_\tau, \widetilde{\tau}\otimes W'_\tau) = \mathrm{Hom}_\Lambda(\widetilde{\tau},\widetilde{\tau})\otimes_R \mathrm{Hom}_R(W_\tau, W'_\tau), \tag{4.5.3.3}$$

using that W_τ and W'_τ are R-free of finite rank and Λ has no effect on them. But $R\hookrightarrow \mathrm{Hom}_\Lambda(\widetilde{\tau},\widetilde{\tau})$ is an isomorphism because $\mathrm{End}_\Lambda(\tau) = k$ and because surjectivity can be checked modulo $\mathfrak{m}_R$ using Lemma 4.5.1.2. As $M_\tau \simeq \widetilde{\tau}\otimes W_\tau$, this implies that

$$\mathrm{Hom}_{\Gamma_\tau}(M_\tau, M'_\tau) = \mathrm{Hom}_\Lambda(M_\tau, M'_\tau)^{T_\tau} = \mathrm{Hom}_\Lambda(\widetilde{\tau}\otimes W_\tau, \widetilde{\tau}\otimes W'_\tau)^{T_\tau} = \mathrm{Hom}_R(W_\tau, W_\tau)^{T_\tau} = \mathrm{Hom}_{T_\tau}(W_\tau, W'_\tau)$$

where T_τ is the image of $T_{L,\tau}$ in Γ_τ. An analogous computation in the case $\tau\in\Sigma_c$ completes the proof. ☐

We can now consider the duality isomorphism $M \simeq M^\vee$. By Lemma 4.5.3.1, this is equivalent to a collection of isomorphisms of $R[T_{L,\tau}]$-modules $\varphi_\tau : W_\tau \simeq W_{\tau^*}^\vee$ for $\tau\in\Sigma_e\cup\Sigma_n$ and an isomorphism of $R[T_{L,\tau\oplus\tau^*}]$-modules $\varphi_\tau : W_{\tau\oplus\tau^*} \simeq W_{\tau\oplus\tau^*}^\vee$ for $\tau\in\Sigma_c$. We analyze the cases separately.

In **Case 1** ($\tau\in\Sigma_n$), note that $\mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau$ is an isotropic subspace of M as $\tau\not\simeq\tau^*$. In particular, the perfect sign-symmetric pairing on $\mathrm{Ind}^{\Gamma_L}\Gamma_{L,\tau}M_\tau \oplus \mathrm{Ind}^{\Gamma_L}\Gamma_{L,\tau^*}M_{\tau^*}$ is equivalent to an isomorphism of $R[\Gamma_L]$-modules

$$\mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau \simeq \left(\mathrm{Ind}_{\Gamma_{L,\tau^*}}^{\Gamma_L} M_{\tau^*}\right)^\vee,$$

which is equivalent to the isomorphism of $R[T_{L,\tau}]$-modules $\varphi_\tau : W_\tau \simeq W_{\tau^*}^\vee$. (Note that the similitude character ν is present in the use of the dual.)

In **Case 2** ($\tau\in\Sigma_e$), the perfect sign-symmetric pairing on $\mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} M_\tau$ is equivalent to an isomorphism $W_\tau\simeq W_\tau^\vee$ of $R[T_{L,\tau}]$-modules. Thus it gives a pairing $\langle,\rangle_{W_\tau}$ on W_τ via

$$\langle w_1, w_2\rangle_{W_\tau} := \varphi_\tau(w_1)(w_2).$$

We claim this pairing is sign-symmetric.

From §4.5.2 we have an isomorphism $\imath : \widetilde{\tau} \simeq \widetilde{\tau}^\vee$ of $R[\Gamma_{L,\tau}]$-modules. Let $\psi : M \to M^\vee$ be the isomorphism of $R[\Gamma_L]$-modules given by $m \mapsto \langle m, - \rangle_M$, and define $\langle v_1, v_2 \rangle_{\widetilde{\tau}} := \imath(v_1)(v_2)$. We have a commutative diagram

$$\begin{array}{ccccc}
\widetilde{\tau} \otimes W_\tau & \xrightarrow{\mathrm{id} \otimes \varphi_\tau} & \widetilde{\tau} \otimes W_\tau^\vee & \xrightarrow{\imath \otimes \mathrm{id}} & \widetilde{\tau}^\vee \otimes W_\tau^\vee \\
\downarrow & & & & \downarrow \\
M_\tau & & \xrightarrow{\quad\quad\psi\quad\quad} & & M_\tau^\vee
\end{array}$$

The commutativity says that for an elementary tensor $m_i = v_i \otimes w_i \in M_\tau = \widetilde{\tau} \otimes W_\tau$ we have

$$\langle m_1, m_2 \rangle_M = \psi(m_1)(m_2) = (\imath(v_1) \otimes \varphi_\tau(w_1))(v_2 \otimes w_2) = \imath(v_1)(v_2) \cdot \varphi_\tau(w_1)(w_2) = \langle v_1, v_2 \rangle_{\widetilde{\tau}} \langle w_1, w_2 \rangle_{W_\tau}. \tag{4.5.3.4}$$

The pairings are perfect and $\langle \cdot, \cdot \rangle_\tau$ is sign-symmetric, so the pairing on W_τ is sign-symmetric if and only if the pairing on M_τ is sign-symmetric.

In **Case 3** ($\tau \in \Sigma_c$), an analogous argument using the isomorphism $\widetilde{\tau \oplus \tau^*} \simeq \widetilde{\tau \oplus \tau^*}^\vee$ of $R[\Gamma_{L,\tau\oplus\tau^*}]$-modules shows that the pairing induced by $\varphi_\tau : W_{\tau\oplus\tau^*} \simeq W_{\tau\oplus\tau^*}^\vee$ is sign-symmetric if and only if the pairing on

$$\mathrm{Ind}_{\Gamma_{L,\tau\oplus\tau^*}}^{\Gamma_L} \left(\widetilde{\tau \oplus \tau^*} \otimes W'_{\tau\oplus\tau^*} \right)$$

induced from the pairing on M is sign-symmetric.

We can now define the minimally ramified deformation condition for $\overline{\rho} : \Gamma_L \to G(k)$, under the continuing assumption that we have extended k so all irreducible representations of Λ_L occurring in V are absolutely irreducible over k. From (4.5.3.2), we obtain a decomposition

$$V = \bigoplus_{\tau \in \Sigma_n} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} \left(\widetilde{\tau} \otimes \overline{W}_\tau \right) \oplus \bigoplus_{\tau \in \Sigma_e} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} \left(\widetilde{\tau} \otimes \overline{W}_\tau \right) \oplus \bigoplus_{\tau \in \Sigma_c} \mathrm{Ind}_{\Gamma_{L,\tau\oplus\tau^*}}^{\Gamma_L} \left(\widetilde{\tau \oplus \tau^*} \otimes \overline{W}_{\tau\oplus\tau^*} \right). \tag{4.5.3.5}$$

where $\overline{W}_\tau$ is a representation of $T_{L,\tau}$ over k. If $\tau \in \Sigma_n$, define $\overline{G}_\tau := \underline{\mathrm{Aut}}(\overline{W}_\tau)$. If $\tau \in \Sigma_e$, there is a sign-symmetric perfect pairing $\langle \cdot, \cdot \rangle_{\overline{W}_\tau}$ on $\overline{W}_\tau$: in that case define $\overline{G}_\tau := \underline{\mathrm{GAut}}(\overline{W}_\tau, \langle \cdot, \cdot \rangle_{\overline{W}_\tau})$. (The notation $\underline{\mathrm{GAut}}$ means automorphisms preserving the pairing up to scalar.) If $\tau \in \Sigma_c$, there is a sign-symmetric perfect pairing on $\overline{W}_{\tau\oplus\tau^*}$: in that case define $\overline{G}_\tau := \underline{\mathrm{GAut}}(\overline{W}_{\tau\oplus\tau^*}, \langle \cdot, \cdot \rangle_{\overline{W}_{\tau\oplus\tau^*}})$. Make a finite extension of k so that all the pairings are split. Lift $\overline{G}_\tau$ to a split reductive group G_τ over $\mathcal{O}$ by lifting the split linear algebra data.

Definition 4.5.3.2. Let $\rho : \Gamma_L \to G(R)$ be a continuous Galois representation lifting $\overline{\rho}$ as above, with associated $R[\Gamma]$-module

$$M = \bigoplus_{\tau \in \Sigma_n} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} \left(\widetilde{\tau} \otimes W_\tau \right) \oplus \bigoplus_{\tau \in \Sigma_e} \mathrm{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L} \left(\widetilde{\tau} \otimes W_\tau \right) \oplus \bigoplus_{\tau \in \Sigma_c} \mathrm{Ind}_{\Gamma_{L,\tau\oplus\tau^*}}^{\Gamma_L} \left(\widetilde{\tau \oplus \tau^*} \otimes W_{\tau\oplus\tau^*} \right).$$

We say that ρ is *minimally ramified* with similitude character ν if each W_τ and $W_{\tau\oplus\tau^*}$ is minimally ramified in the sense of Definition 4.4.2.1 as a representation of $T_{L,\tau}$ or $T_{L,\tau\oplus\tau^*}$ valued in the group G_τ. (Note that defining the minimally ramified deformation condition as in §4.4 may require an additional étale local extension of $\mathcal{O}$, which as always is harmless for applications.)

Let $D_{\overline{\rho}}^{\mathrm{m.r.},\nu}$ denote the deformation functor for $\overline{\rho}$ with specified similitude character ν, and $\mathcal{D}_{G_\tau}^{\mathrm{m.r.},\nu}$ (respectively $D_{G_\tau}^{\mathrm{m.r.},\nu}$) denote the deformation functor for $\overline{W}_\tau$ or $\overline{W}_{\tau\oplus\tau^*}$ viewed as a representation valued in G_τ (respectively with specified similitude character ν). In particular, letting $r = \dim \overline{W}_\tau$ (or $\dim \overline{W}_{\tau\oplus\tau^*}$ when $\tau \in \Sigma_c$), we have that the adjoint representation $\mathrm{ad}\, \overline{W}_\tau$ is the Lie algebra of $\overline{G}_\tau$, which is the Lie algebra of GSp_r or GO_r when $\tau \in \Sigma_e$ or Σ_c, and the Lie algebra of GL_r when $\tau \in \Sigma_n$. Let Σ'_n consist of one representative for each pair of representations $\tau, \tau^* \in \Sigma_n$.

Proposition 4.5.3.3. *The natural map*

$$D_{\overline{\rho}}^{\mathrm{m.r.}\ \nu}(R) \to \prod_{\tau \in \Sigma'_n} D_{G_\tau}^{\mathrm{m.r.}}(R) \times \prod_{\tau \in \Sigma_e} D_{G_\tau}^{\mathrm{m.r.}\ \nu}(R) \times \prod_{\tau \in \Sigma_c} D_{G_\tau}^{\mathrm{m.r.}\ \nu}(R)$$

is an isomorphism.

Proof. This expresses the decomposition obtained in this section: given a lift ρ of $\overline{\rho}$, we obtain a decomposition of M as in Definition 4.5.3.2. Our analysis with pairings shows that when $\tau \in \Sigma_e$, W_τ is a deformation of $\overline{W}_\tau$ together with its sign-symmetric perfect pairing. Likewise, when $\tau \in \Sigma_c$ we know that $W_{\tau\oplus\tau^*}$ is a deformation of $\overline{W}_{\tau\oplus\tau^*}$ together with its pairing. When $\tau \in \Sigma_n$, we know $W_\tau \simeq W_{\tau^*}^\vee$. This gives the natural map: to $\rho \in D_{\overline{\rho}}^{\text{m.r.}\ \nu}(R)$ associate the collection of the W_τ for $\tau \in \Sigma_e \cup \Sigma_c \cup \Sigma_n'$.

Conversely, given W_τ for $\tau \in \Sigma_e \cup \Sigma_c \cup \Sigma_n'$, and defining $W_{\tau^*} := W_\tau^\vee$ for $\tau \in \Sigma_n'$ we can define a lift

$$M := \bigoplus_{\tau\in\Sigma_n} \operatorname{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau}\otimes W_\tau) \oplus \bigoplus_{\tau\in\Sigma_e} \operatorname{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau}\otimes W_\tau) \oplus \bigoplus_{\tau\in\Sigma_c} \operatorname{Ind}_{\Gamma_{L,\tau\oplus\tau^*}}^{\Gamma_L}\left(\widetilde{\tau\oplus\tau^*}\otimes W_{\tau\oplus\tau^*}\right).$$

as in (4.5.1.1). (Note that the groups $\Gamma_{L,\tau}$ depend only on the fixed residual representation V.) For $\tau \in \Sigma_e$, the sign-symmetric perfect pairing on the lift W_τ gives an isomorphism $\varphi_\tau : W_\tau \simeq W_\tau^\vee$ of $R[T_{L,\tau}]$-modules, which gives a sign-symmetric pairing on $\operatorname{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau}\otimes W_\tau)$ (using (4.5.3.4)). Likewise, $\tau\in\Sigma_c$ the sign-symmetric pairing on $W_{\tau\oplus\tau^*}$ gives one on $\operatorname{Ind}_{\Gamma_{L,\tau\oplus\tau^*}}^{\Gamma_L}\left(\widetilde{\tau\oplus\tau^*}\otimes W_{\tau\oplus\tau^*}\right)$. For $\tau \in \Sigma_n$, we obtain an isomorphism $\varphi_\tau : W_\tau \simeq W_{\tau^*}^\vee$ of $R[T_{L,\tau}]$-modules and hence a sign-symmetric perfect pairing on $(\widetilde{\tau}\otimes W_\tau)\oplus(\widetilde{\tau}^\vee\otimes W_{\tau^*})$ which gives one on $\operatorname{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau}\otimes W_\tau)\oplus\operatorname{Ind}_{\Gamma_{L,\tau}}^{\Gamma_L}(\widetilde{\tau}^\vee\otimes W_{\tau^*})$. Putting these together, we obtain a sign-symmetric pairing on M; the action of Γ_L preserves it up to scalar, giving a continuous homomorphism $\rho : \Gamma_L \to G(R)$.

Finally, we claim that these constructions are compatible with strict equivalence of lifts, giving an identification of the deformation functors. For $g \in \widehat{G}(R)$, decompose the g-conjugate Γ_L-representation M^g according to (4.5.3.2). As g reduces to the identity, it must respect the decomposition into τ-isotypic pieces, so gives automorphisms $g_\tau \in \operatorname{Aut}(W_\tau)$ and $g_\tau \in \operatorname{Aut}(W_{\tau\oplus\tau^*})$. If $\tau\in\Sigma_e$ or Σ_c, as g is compatible with the pairing on M we see g_τ is compatible with the pairing as well. For $\tau \in \Sigma_e$, the g_τ-conjugate $T_{L,\tau}$-representation $W_\tau^{g_\tau}$ is minimally ramified as minimally ramified lifts of $\overline{W}_\tau$ for the group $T_{L,\tau}$ are a deformation condition, and likewise for $\tau\in\Sigma_c$ and $\tau\in\Sigma_n'$.

Conversely, given $g_\tau \in \operatorname{Aut}(W_\tau)$ reducing to the identity (compatible with the pairing on W_τ or $W_{\tau\oplus\tau^*}$ if there is one), using (4.5.1.1) and acting on each piece we obtain a lift of the form M^g for $g\in\widehat{G}(R)$. Thus the identification is compatible with strict equivalence. ☐

Corollary 4.5.3.4. *The minimally ramified deformation condition with fixed similitude character is liftable. The dimension of the tangent space is* $h^0(\Gamma_L, \operatorname{ad}^0(\overline{\rho}))$.

Proof. Liftability is a consequence of Proposition 4.5.3.3 and the smoothness of the framed minimally ramified deformation ring for representations of $T_{L,\tau}$ (Proposition 4.4.2.3 and Corollary 4.4.2.5). Recall that the dimension of the tangent space of a deformation condition of a representation $\theta : T_{L,\tau} \to G_\tau(k)$ is the dimension of the tangent space of the framed deformation ring minus the relative dimension of G_τ plus the dimension of $H^0(T_{L,\tau}, \operatorname{ad}\theta)$ (see Remark 2.2.2.6). By Corollary 4.4.2.5, for $\tau\in\Sigma_e$ the dimension of the tangent space of $D_{G_\tau}^{\text{m.r.},\nu}$ is $h^0(T_{L,\tau}, \operatorname{ad}\overline{W}_\tau) - 1 = h^0(T_{L,\tau}, \operatorname{ad}^0\overline{W}_\tau)$, and for $\tau\in\Sigma_c$ the dimension is $h^0(T_{L,\tau\oplus\tau^*}, \operatorname{ad}\overline{W}_{\tau\oplus\tau^*}) - 1 = h^0(T_{L,\tau\oplus\tau^*}, \operatorname{ad}^0\overline{W}_{\tau\oplus\tau^*})$. For $\tau\in\Sigma_n'$, by Proposition 4.4.2.3 the dimension of the tangent space of $D_{G_\tau}^{\text{m.r.}}$ is $h^0(T_{L,\tau}, \operatorname{ad}\overline{W}_\tau)$. Using Proposition 4.5.3.3, we see that the dimension of the tangent space of the minimally ramified deformation condition is

$$\sum_{\tau\in\Sigma_e} h^0(T_{L,\tau}, \operatorname{ad}^0\overline{W}_\tau) + \sum_{\tau\in\Sigma_c} h^0(T_{L,\tau\oplus\tau^*}, \operatorname{ad}^0\overline{W}_{\tau\oplus\tau^*}) + \sum_{\tau\in\Sigma_n'} h^0(T_{L,\tau}, \operatorname{ad}\overline{W}_\tau).$$

It remains to identify this quantity with $h^0(\Gamma_L, \operatorname{ad}^0(\overline{\rho}))$. Using Lemma 4.5.3.1

$$\begin{aligned} H^0(\Gamma_L, \operatorname{End}(V)) = \operatorname{Hom}_{k[\Gamma_L]}(V,V) &= \bigoplus_{\tau\in\Sigma_e\cup\Sigma_n} \operatorname{Hom}_{T_{L,\tau}}(\overline{W}_\tau, \overline{W}_\tau) \oplus \bigoplus_{\tau\in\Sigma_c} \operatorname{Hom}_{T_{L,\tau\oplus\tau^*}}(\overline{W}_{\tau\oplus\tau^*}, \overline{W}_{\tau\oplus\tau^*}) \\ &= \bigoplus_{\tau\in\Sigma_e\cup\Sigma_n} H^0(T_{L,\tau}, \operatorname{End}(\overline{W}_\tau)) \oplus \bigoplus_{\tau\in\Sigma_c} H^0(T_{L,\tau\oplus\tau^*}, \operatorname{End}(\overline{W}_{\tau\oplus\tau^*})). \end{aligned}$$

We are interested in $H^0(\Gamma_L, \operatorname{ad}^0(\overline{\rho}))$: the elements $\psi \in H^0(\Gamma_L, \operatorname{End}(V))$ compatible with the pairing on V in the sense that for $v, v' \in V$

$$\langle \psi v, \psi v'\rangle = \langle v, v'\rangle.$$

The pairing on $V_\tau = \tau \otimes \overline{W}_\tau$ is induced by the pairings on $\overline{W}_\tau$ and τ when $\tau \in \Sigma_e$, and is induced by the pairings on $\overline{W}_{\tau\oplus\tau^*}$ and $\tau \oplus \tau^*$ when $\tau \in \Sigma_c$. When $\tau \in \Sigma'_n$, the pairing on $V_\tau \oplus V_{\tau^*}$ comes from the $\Gamma_{L,\tau}$-isomorphism $V_\tau \simeq V_{\tau^*}^\vee$ which in turn comes from the $T_{L,\tau}$-isomorphism $\overline{W}_\tau \simeq \overline{W}_{\tau^*}^\vee$. So ψ is compatible with the pairing if and only if

- when $\tau \in \Sigma_e$, the associated $\psi_\tau \in H^0(T_{L,\tau}, \mathrm{End}(\overline{W}_\tau))$ is compatible with the pairing on $\overline{W}_\tau$;
- when $\tau \in \Sigma_c$, the associated $\psi_\tau \in H^0(T_{L,\tau\oplus\tau^*}, \mathrm{End}(\overline{W}_{\tau\oplus\tau^*})$ is compatible with the pairing on $\overline{W}_{\tau\oplus\tau^*}$;
- when $\tau \in \Sigma'_n$, the associated ψ_τ and ψ_{τ^*} are identified by duality and the isomorphism $\overline{W}_\tau \simeq \overline{W}_{\tau^*}^\vee$.

In the first two cases, $\mathrm{ad}^0\,\overline{W}_\tau$ and $\mathrm{ad}^0\,\overline{W}_{\tau\oplus\tau^*}$ are the symplectic or orthogonal Lie algebra, consisting exactly of endomorphisms compatible with the pairing on $\overline{W}_\tau$. In the third, we just choose one of ψ_τ and ψ_{τ^*} without restriction, which determines the other. Thus we see

$$H^0(\Gamma_L, \mathrm{ad}^0(\overline{\rho})) = \bigoplus_{\tau\in\Sigma_e} H^0(T_{L,\tau}, \mathrm{ad}^0\,\overline{W}_\tau) \oplus \bigoplus_{\tau\in\Sigma_c} H^0(T_{L,\tau\oplus\tau^*}, \mathrm{ad}^0\,\overline{W}_{\tau\oplus\tau^*}) \oplus \bigoplus_{\tau\in\Sigma'_n} H^0(T_{L,\tau}, \mathrm{ad}\,\overline{W}_\tau). \qquad \square$$

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