INTERSECTION THEORY IN ALGEBRAIC GEOMETRY: COUNTING LINES IN THREE DIMENSIONAL SPACE

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Intersection theory allows you to count. How many lines lie in two quadric hypersurfaces in $\mathbb{P}^4$? How many lines lie in a plane and pass through a general point in $\mathbb{P}^3$? How many conics in the plane are tangent to each of five general conics? The way to approach these problems is to determine information about a larger variety (the class of lines lying in one quadric hypersurface, the class of conics tangent to one conic in these cases) and then intersect these varieties to get a finite collection of points.

The fundamental ingredient in intersection theory is the choice of an appropriate parameter space. For points, the projective space $\mathbb{P}^n$ is a good parameter space. For lines, the parameter space is a Grassmanian. The calculations take place in the Chow ring, where the multiplication operation corresponds to intersection of sub-varieties.

Unfortunately, all of this becomes quiet technical. To deal with much of this rigorously, one needs to understand scheme theory, a very non-trivial area of algebraic geometry. Therefore, most of the following will be wrong. But it's not very wrong. It will illustrate the ideas behind intersection theory, and show why it is a neat and powerful technique in reducing hard problems in geometry to easier algebra.

1. Complex Projective Space

PODASIP: In a plane, any two distinct lines meet in a unique point.

It almost goes without saying that to count the number of points of intersection in a general manner, we need to get a consistent answer. If two distinct lines meet in a unique point, well and good. But if they sometimes do not meet at all (consider $x = 0$ and $x = 1$ in $\mathbb{R}^2$) and sometime meet in one point, counting will be near impossible. Although one could argue that parallel lines are a tiny minority of all lines and so we should ignore this answer, this causes too many problems later. Our solution will be to add points "at infinity".

But first, we have another problem. Some systems of equations have complex roots. For example, if we try to find the intersection of the parabola $y - x^2 = 0$ with the line $y = -1$, there are no real points of intersection because the solutions to $x^2 = -1$ are $(x, y) = (\pm i, -1)$. On the other hand, intersecting with the line $y = 1$ gives two solutions. To resolve this issue, we use the complex numbers instead of the real numbers. In the complex numbers, unless a line is tangent to this parabola there are two points of intersections which we can determine algebraically. Algebraic geometry attempts to do hard geometry by doing algebra with polynomials. Since every polynomial with complex coefficients has complex roots, by choosing the complex numbers we get the algebra to work.

Remark 1. What happens if the line is tangent to the parabola? For example, try intersecting $y - x^2 = 0$ with $y = 0$. The only point of intersection is $(0, 0)$. However, tangency happens only for very special lines, and it is fine to ignore them. In fact, the convention that $x^2 = 0$

Date: July 6, 2010.

has a double root at $x = 0$ suggests we should count that point of intersection twice. This is the tip of a theory that assigns multiplicities to points of intersection to make our counts always correct, at the expense of vast complications. It is also possible to ignore these exceptional cases by insisting on a technical condition called generic transversality, as hinted at near the end of Section 3.

We now deal with adding points at infinity.

Definition 2. Define two nonzero points $(x_0, x_1, x_2, \ldots, x_n)$ and $(y_0, y_1, y_2, \ldots, y_n)$ in $\mathbb{C}^{n+1}$ to be equivalent if there is a constant $\lambda \in \mathbb{C}^\times$ such that $x_i = \lambda y_i$ for $0 \leq i \leq n$. Group the nonzero points of $\mathbb{C}^{n+1}$ together into classes of equivalent points. Denote the set of classes by $\mathbb{P}^n$. It is known as complex projective space.

Let $[x_0, x_1, x_2, \ldots, x_n]$ denote the equivalence class in $\mathbb{C}^{n+1}$ of all points that are a non-zero multiple of $(x_0, x_1, x_2, \ldots, x_n)$.

At first, this may seem a strange definition. However, consider the following set:

$$A := \{[x_0, x_1, x_2, x_3] \in \mathbb{P}^3 : x_3 \neq 0\}$$

Since the last coordinate is nonzero, $[x_0, x_1, x_2, x_3] = [x_0 x_3^{-1}, x_1 x_3^{-1}, x_2 x^{-3}, 1]$. Since every class has a unique representative with the last coordinate 1, the set A is in bijection with $\mathbb{C}^3$, where

$$[x_0, x_1, x_2, 1] \to (x_0, x_1, x_2) \in \mathbb{C}^3.$$

So we have a copy of three dimensional space inside $\mathbb{P}^3$, along with the other stuff

$$B := \{[x_0, x_1, x_2, 0]\} \subset \mathbb{P}^3.$$

This other stuff is what we call the points at infinity.

Remark 3. You can work out that the points at infinity are a projective plane $\mathbb{P}^2$, which in turn consists of a copy of $\mathbb{C}^2$ and a projective line, which in turn gives a copy of $\mathbb{C}^2$, a copy of $\mathbb{C}^1$, and a point.

Now, what is a plane in $\mathbb{P}^3$? Well, on the copy of $\mathbb{C}^3$, I know what a plane is. For example, it is $x_0 + x_1 + x_2 - 1 = 0$. Can I extend this equation to $\mathbb{P}^3$? No, since if $x_0 + x_1 + x_2 - 1 = 0$, the point $[2x_0, 2x_1, 2x_2, 2x_3]$ is a representative for the same class, and $2x_0 + 2x_1 + 2x_2 - 1 = 1$.

However, remember that on the copy of $\mathbb{C}^3$, I scaled so $x_3 = 1$. If I replace the 1 by x_3, I get $x_0 + x_1 + x_2 - x_3 = 0$. This equation is homogeneous (each term of the sum has the same degree), so if I scale all my variables by λ I simply get $\lambda x_0 + \lambda x_1 + \lambda x_2 + \lambda x_3 - \lambda x_3 = \lambda \cdot 0 = 0$. This is independent of the choice of representative for a class in $\mathbb{P}^3$, so we are good. This is an example of a process called homogenization.

Definition 4. Let $\{f_i\}$ be a collection of homogeneous polynomials of $n + 1$ variables. The set of points in $\mathbb{P}^n$ where all of the f_i vanish is called a projective variety.

Example 5. A surface in $\mathbb{P}^3$ is the zero locus of a single homogeneous polynomial. The simplest are the zero loci of degree 1 and degree 2 polynomials, which are called a plane and a quadric surface respectively.

Example 6. A projective line is the zero locus of two degree one polynomials: it is the intersection of two planes containing the line. It is the set of $[x_0, x_1, x_2, x_3]$ such that

$$a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \quad \text{and} \quad b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$

for some complex a_i, b_i that are independent.

Again, note that on the copy of $\mathbb{C}^3$ where every point has a representative with $x_3 = 1$, this definition is $a_0x_0 + a_1x_1 + a_2x_2 + a_3 = 0$ and $b_0x_0 + b_1x_1 + b_2x_2 + b_3 = 0$, the normal definition of a line in three dimensional space.

We can now look at the question of whether any two lines in a plane meet. In the plane $x_0 = 0$, consider the lines $x_1 = 0$ and $x_1 = x_3$, which is an example of the situation which motivated the introduction of projective space. (Remember $x_3 = 1$ in the patch of $\mathbb{P}^3$ corresponding to finite points.) These lines are "parallel" in the plane, but they meet at $[0, 0, 1, 0]$, one of the points at infinity.

Example 7. Consider the map $\mathbb{P}^1 \to \mathbb{P}^3$ defined by sending $[s, t] \to [s^3, s^2t, st^2, t^3]$. Its image in $\mathbb{P}^3$ is called a twisted cubic curve. It is a projective variety. To see this, note that any point in the image satisfies

$$x_0x_2 - x_1^2 = 0 \quad \text{and} \quad x_0x_3 - x_1x_2 = 0 \quad \text{and} \quad x_1x_3 - x_2^2 = 0$$

because

$$s^3 \cdot st^2 - (s^2t)^2 = 0 \quad \text{and} \quad s^3t^3 - (s^2t) \cdot (st^2) = 0 \quad \text{and} \quad s^2t \cdot t^3 - (st^2)^2 = 0.$$

Therefore any point of the image lies on these three quadric surfaces. You can check that any point on the intersection of all three surfaces lies in the image.

There are many other twisted cubic curves: simply move it around any automorphism of three dimensional space, for example by reflecting, translating, or rotating it.[1]

We can now pose our two keynote questions. The term general means that the answer should hold for "most" configurations in a way that will be defined later.

(1) Given four general lines in $\mathbb{P}^3$, how many lines meet all four?
(2) How many lines in $\mathbb{P}^3$ are secant to (intersect at least twice) two general twisted cubic curves?

2. Grassmanians

To count lines, we need some way to describe the space of all lines in $\mathbb{P}^3$. The way to do this is through Grassmanians, an example of a more general concept called parameter spaces.

A line in $\mathbb{P}^3$ is uniquely determined by two points on it. Let $x = [x_0, x_1, x_2, x_3]$ and $y = [y_0, y_1, y_2, y_3]$ be the coordinates of two points on the line. Map it to the point

$$[x_0y_1 - x_1y_0, x_0y_2 - x_2y_0, x_0y_3 - x_3y_0, x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2] \in \mathbb{P}^5.$$

Viewing this as a map $\wedge : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}^6$, it is simply a description in coordinates of the wedge product $V \times V \xrightarrow{\wedge} \Lambda^2(V)$. Either directly or through the properties of the wedge product, one can verify that this map is linear in both variables and that if $x = y$ then the image is 0. Therefore, if we pick different points on the line, which can be expressed as linear combinations of x and y, then $(ax+by)\wedge(cx+dy) = (ad-bc)x\wedge y$ which is a multiple of our original result. Since the range is a projective space, these determine the same point in $\mathbb{P}^5$ and hence this map is well defined. Denoting the coordinates of $\mathbb{P}^5$ by $z_{i,j}$ (with the indices corresponding to the wedge product map), algebra shows that $z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23} = 0$ and that any point of $\mathbb{P}^5$ satisfying this relation is in the image. Therefore the image is a

[1]Formally, act on $\mathbb{P}^3$ by $\mathrm{PGL}_4(\mathbb{C})$

projective variety. It is called the Grassmanian $\mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$. As an exercise, show that the map is injective, so $\mathbb{G}(1,3)$ is a parameter space for lines in $\mathbb{P}^3$.

We now describe some special sub-varieties called Schubert cycles in the Grassmanian $\mathbb{G}(1,3)$. They depend on the choice of a flag in $\mathbb{P}^3$, which is the choice of a point p, a line L containing the point, and a plane Λ containing the line. It is straightforward to show these are projective varieties using more techniques from algebraic geometry like incidence correspondences.

- $\Sigma_{2,2}$ is all lines in the Grassmanian contained in L, so is just L. Note that $\Sigma_{2,2}$ is a *point* in $\mathbb{G}(1,3) \subset \mathbb{P}^5$. It is zero dimensional.
- $\Sigma_{2,1}$ is all lines though a p lying in Λ. It is one dimensional.
- $\Sigma_{1,1}$ is all lines lying in Λ. It is two dimensional.
- Σ_2 is all lines passing through the point p. It is also two dimensional.
- Σ_1 is all lines intersecting the line L. It is three dimensional.

Note that the sum of the subscripts is 4 minus the dimension of the Schubert cycle.

If it is necessary to indicate which flag these are based on, the relevant point, line, or plane will be indicated in brackets: $\Sigma_1(L_2)$ would be lines meeting the line L_2.

3. Intersection Theory and the Chow Ring

To answer these questions about intersections, we follow the fundamental philosophy of algebraic geometry and reduce the question to algebra. The first notion to consider is rational equivalence. We will use this to define the Chow Ring and intersection products.

Definition 8. A cycle is a finite $\mathbb{Z}$-linear combination of varieties. Two cycles $a_1V_1 + a_2V_2 + \ldots + a_nV_n$ and $b_1W_1 + \ldots + b_nW_n$ with $a_i > 0$ and $b_i > 0$ are rationally equivalent if there exists a smooth variety

$$Z \subset \mathbb{P}^1 \times Y$$

such that $\{(t,x) \in Z : t = [0,1]\}$ is the union of a_i copies of V_i and $\{(t,x) \in Z : t = [1,1]\}$ is the union of b_i copies of W_i for all i.

Remark 9. How do you count how many copies of each variety should be included? It's a very difficult question. For example, consider the variety

$$\{[s,t] \times [x_0, x_1, x_2] : x_0x_2 - x_1^2 = 0 \quad \text{and} \quad tx_0 + (s-t)x_2\} \subset \mathbb{P}^1 \times \mathbb{P}^3.$$

When $[s,t] = [0,1]$, $x_0x_2 - x_1^2 = 0$ and $x_0 - x_2 = 0$, which are the projective equations for the parabola $y = x^2$ and line $y = 1$. There are two points of intersection. When $[s,t] = [1,1]$, $x_0x_2 - x_1^2 = 0$ and $x_0 = 0$, which are the projective equations for the parabola $y = x^2$ and line $y = 0$, there is a single point of intersection. One point should not be equivalent to two points. The problem is that the parabola has a double root. Again, a theory of multiplicities can solve this problem.

If our cycles have just one component (X_0 and X_1 respectively), what this means is that $X_s = \{(t,x) \in Z : t = [s,1]\} \subset \mathbb{P}^n$ interpolates smoothly between X_0 and X_1.

Example 10. Any two lines in $\mathbb{P}^3$ are rationally equivalent. To see this, let f_1 and g_1 be the equations of two distinct planes containing the first line, and f_2 and g_2 be two distinct planes containing the second line. The $x \in \mathbb{P}^3$ such that $f_1(x) = g_1(x) = 0$ is the first line, and the $x \in \mathbb{P}^3$ such that $f_2(x) = g_2(x) = 0$ are the second line. Consider the variety in $\mathbb{P}^1 \times \mathbb{P}^3$ defined by

$$\{([s,t],x) : sf_1(x) + (t-s)f_2(x) = 0 \quad \text{and} \quad sg_1(x) + (t-s)g_2(x) = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3.$$

Note this is homogeneous in both $[s,t]$ and x. When $[s,t] = [0,1]$, the f_1 and g_1 terms disappear, so the zero locus is just the zero locus of f_2 and g_2, the second line. When $[s,t] = [1,1]$, the f_2 and g_2 terms disappear, so the zero locus is the first line. Thus the two lines are rationally equivalent.

Now we can define the Chow ring.

Definition 11. Let X be a projective variety. The Chow Ring $A^*(X)$ is defined to be the set of all cycles modulo the equivalence relation of rational equivalence. Cycles of the form

$$a_1[V_1] + a_2[V_2] + \ldots + a_n[V_n]$$

where $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ are added by combining terms with the same $[V_i]$ by adding the integers. Multiplication is defined for single varieties by intersecting: $[V] \cdot [W] = [V \cap W]$. To form a ring, we extend this definition so that

$$(a_1[V_1] + a_2[V_2] + \ldots + a_n[V_n])(b_1[V_1] + b_2[V_2] + \ldots + b_n[V_n]) = \sum_{i,j} a_i b_j [V_i] \cdot [V_j]$$

If a variety V is made up of two components W and W', we further stipulate that $[V] = [W] + [W']$.

Remark 12. This definition is false for the very simple reason that if $V_1 \sim V_2$ then $V_1 \cap W$ need not be rationally equivalent to $V_2 \cap W$. For example, two rationally equivalent lines could be intersected with a parabola, but one could be a tangent line. One point is not rationally equivalent to two points. Again, this could be salvaged by developing machinery to detect multiplicities.

Despite these problems, there is no difficulty in calculating with the Chow ring. When intersecting $[V_1]$ and $[V_2]$, some choices of representatives may give the wrong answer. There is a technical condition called generic transversality that guarantees that intersecting the varieties will give the correct answer, and a lemma called the moving lemma that says it always possible to pick representatives for the equivalence classes so that the intersection is generically transverse. Essentially, all one does is slightly move both of the varieties, which causes the exceptional intersection to go away. In our example, slightly disturbing the parabola will cause the line to no longer be tangent and have the correct number of intersection points.

4. The Chow Ring of $\mathbb{G}(1,3)$

In order to answer the keynote questions, we need to understand the Chow ring of $\mathbb{G}(1,3)$. The hardest part is finding a complete set of representatives for rational equivalence classes of sub-varieties. Because of the decomposition into Schubert cycles, a general theorem about affine stratifications implies that the Chow ring is of the form

$$\mathbb{Z}[[\mathbb{G}(1,3)], [\Sigma_1], [\Sigma_2], [\Sigma_{1,1}], [\Sigma_{2,1}], [\Sigma_{2,2}]]$$

We will use lower case σ to denote the classes: $\sigma_2 = [\Sigma_2]$ for example.

What remains to do is to determine the intersection product. In order to avoid problems with "bad" intersections, we will use different flags: as long as the flags are general, and assumed to have no unexpected intersections (like the two lines intersecting, which does not happen for most pairs of lines in $\mathbb{P}^3$) we will be fine.[2] Denote the flags by $p_1 \subset L_1 \subset \Lambda_1$ and $p_2 \subset L_2 \subset \Lambda_2$. An additional fact simplifies the calculations: we know that when intersecting k and l dimensional spaces in the 4 dimensional Grassmanian, the expected dimension of the intersection is $l+k-4$. The following is a partial list of the intersection products, illustrating all the relevant techniques.

- First, note that $[\mathbb{G}(1,3)]$ is the multiplicative identity, because intersecting with the entire space doesn't change anything.
- To calculate $\sigma_2\sigma_{2,1}$ and many other products, simply note that the expected dimension of their intersection is less than 0, so the intersection is empty for general flags.
- To calculate $\sigma_1\sigma_{1,1}$, pick two flags. Σ_1 is the lines intersecting a given line L_1, and $\Sigma_{1,1}$ is the lines lying in a plane Λ_2. L_1 and Λ_2 intersect in a point p, so lines in $\Sigma_1(L_1) \cap \Sigma_{1,1}(\Lambda_2)$ must pass through p and be contained in Λ_2. Thus $\sigma_1\sigma_{1,1} = \sigma_{2,1}$.
- Likewise, $\sigma_2\sigma_1 = \sigma_{2,1}$ because the set of lines passing through a point p_1 and intersecting a line L_2 are all the lines lying in the plane that L_2 and p_1 determine. (Remember that this is a projective plane, so there are no parallel lines.)
- $\sigma_{1,1}^2 = \sigma_{2,2}$ because there is a unique plane lying in planes Λ_1 and Λ_2: the line $\Lambda_1 \cap \Lambda_2$.
- $\sigma_2^2 = \sigma_{2,2}$ as well, because there is a unique line passing through the points p_1 and p_2.
- $\sigma_{1,1}\sigma_2$ is the class of the set of lines that lie in a plane Λ_1 and pass through a point p_2. For a general choice of Λ_1, it will not contain p_2, so the intersection is empty. Therefore $\sigma_{1,1}\sigma_2 = 0$.
- σ_1^2 is the set of lines intersecting two lines L_1 and L_2. This is not a Schubert cycle. In general, the lines L_1 and L_2 will not intersect. If they meet at p, then the set of lines intersecting both L_1 and L_2 would be the all lines that either lie in the plane L_1 and L_2 determine or that pass through the point p. Since we need the intersection to be general in order to avoid problems, this isn't good enough to calculate the intersection product. However, a general configuration is rationally equivalent to this special configuration where we can calculate the intersection. To see this, let L_2' be a line meeting L_1. By an earlier example, we know there is a rational equivalence between L_2 and L_2'. Denote the intermediate line at time t by $L_2(t)$. Then $L_2(0) = L_2$ and $L_2(1) = L_2'$. Then
$$\{(t, \Sigma_1(L_1) \cap \Sigma_1(L_2(t)))\} \subset \mathbb{P}^1 \times \mathbb{G}(1,3)$$
gives a rational equivalence between $\Sigma_1(L_1) \cap \Sigma_1(L_2)$ and $\Sigma_1(L_1) \cap \Sigma_1(L_2')$. We know the second intersection is the union of two components, $\Sigma_{1,1}(\overline{L_1, L_2'})$ and $\Sigma_2(L_1 \cap L_2')$. Thus $\sigma_1^2 = \sigma_{1,1} + \sigma_2$.

5. Four Lines in $\mathbb{P}^3$

With all of this machinery set up, the geometric problem of determining how many lines in $\mathbb{P}^3$ intersect each of four general lines L_1, L_2, L_3, L_4 is almost trivial. The set of lines intersecting L_i is $\Sigma_1(L_i)$. Since the lines are general, the intersection of the $\Sigma_1(L_i)$ will give the correct answer. But in the Chow ring the class of the set of lines intersecting L_i is σ_1.

[2]Formally, $\mathrm{PGL}_4(\mathbb{C})$ acts on $\mathbb{P}^3$ and is transitive on flags. Kleinman's transversality theorem then says that a Schubert cycle for the general flag is generically transverse to a Schubert cycle arising from a fixed flag, and hence that the intersection product can be calculated by intersecting the varieties.

To intersect all 4, I just need to calculate σ_1^4. Using the previously calculated relations from the last section, we get

$$\begin{aligned}\sigma_1^4 &= (\sigma_1^2)^2 = (\sigma_2 + \sigma_{1,1})^2 \\ &= \sigma_2^2 + 2\sigma_2\sigma_{1,1} + \sigma_{1,1}^2 \\ &= \sigma_{2,2} + 0 + \sigma_{2,2} = 2\sigma_{2,2}\end{aligned}$$

Now $\sigma_{2,2}$ is a point in the Grassmanian, which corresponds to one line. Therefore there are 2 lines intersecting four general lines in $\mathbb{P}^3$.

6. Secant lines to Two Twisted Cubic Curves

Finding out how many lines are secant to two twisted cubic curves is also fairly easy with our algebraic machinery. The first step is to find the class in the Chow ring of the set of lines secant to one twisted cubic. Picking a point on the cubic, there is a one dimensional family of secant lines passing through it, corresponding to picking a second point on the one dimensional curve. Therefore there is a two dimensional family of secant lines. Thus the class of secant lines is some linear combination of σ_2 and $\sigma_{1,1}$ since these are the only classes of the correct dimension.

Suppose the class of the secant lines is $a\sigma_2 + b\sigma_{1,1}$. If we intersect with $\Sigma_{1,1}$, I get

$$\sigma_{1,1}(a\sigma_2 + b\sigma_{1,1}) = b\sigma_{2,2}.$$

On the other hand, the secant lines to a twisted cubic that also are in $\Sigma_{1,1}$ (lie in a plane) are the lines joining two of the points where the twisted cubic intersects the plane. If the plane has equation $a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0$, then intersecting with the twisted cubic gives

$$a_0s^3 + a_1s^2t + a_2st^2 + a_3t^3 = 0$$

I may either take $t = 1$, which gives a cubic equation in s which (for a general plane) has three solutions, or $t = 0$ and $s = 1$, which has no solutions for a general plane. Thus there are three points of intersection. The secant lines that arise are those joining two of these three points. There are $\binom{3}{2} = 3$ such lines. Thus $b = 3$.

If I intersect with σ_2, I get

$$\sigma_2(a\sigma_2 + 3\sigma_{1,1}) = a\sigma_{2,2}.$$

How many secant lines pass through a point p? Consider projecting from the point p to a plane: the twisted cubic is mapped to a curve in the plane. If there is a secant line passing through p, two points on the twisted cubic are projected to the same point in the plane, causing the plane curve to intersect itself.

I can calculate this directly. Take $p = [1, 1, 1, 0]$ and project to the plane where $x_2 = 0$. The point on the line spanned by $[1, 1, 1, 0]$ and $[s^3, s^2t, st^2, t^3]$ which lies on the plane is $[s^3 - st^2, s^2t - st^2, 0, t^3]$. Therefore the image of the twisted cubic in the plane is the curve $[s, t] \to [s^3 - st^2, s^2t - st^2, 0, t^3]$. The only time the last coordinate is 0 is when $t = 0$, so there is no self intersection when $t = 0$. For other values of $[s, t]$, since t is nonzero I may scale it so $t = 1$. Then I am looking for two different values of s for which $[s^3 - s, s^2 - s, 0, 1]$ are equal. Suppose $s_1^3 - s_1 = s_2^3 - s_2$ and $s_1^2 - s_1 = s_2^2 - s_2$. Multiplying the second by $s_1 + 1$, I get

$$s_2^3 - s_2 = s_1^3 - s_1 = (s_1 + 1)(s_2^2 - s_2)$$

Either $s_2^2 - s_2 = 0$ or $s_2 + 1 = s_1 + 1$. The second case has $s_2 = s_1$ so is not a point of intersection. The first case forces $s_2 = 0, 1$ and $s_1 = 0, 1$. Thus the images of $[0, 1]$ and $[1, 1]$ agree, and there is a unique point of self-intersection. Thus there is a unique secant line to the curve passing through the point p. This implies there is one line in the secant variety intersected with Σ_2, and hence $a = 1$.

We finally know that the class of the secant variety to a twisted cubic is $\sigma_2 + 3\sigma_{1,1}$. For two general cubics, the secant varieties will intersect transversely so that the number of lines in their intersection is the number predicted by calculating $(\sigma_2 + 3\sigma_{1,1})^2$: this is

$$\begin{aligned}(\sigma_2 + 3\sigma_{1,1})^2 &= \sigma_2^2 + 6\sigma_2\sigma_{1,1} + 9\sigma_{1,1}^2 \\ &= \sigma_{2,2} + 0 + 9\sigma_{2,2} \\ &= 10\sigma_{2,2}\end{aligned}$$

Thus there are 10 lines in $\mathbb{P}^3$ secant to two general twisted cubic curves.

7. Acknowledgments and References

I learned this material from Joe Harris, who is writing a book with David Eisenbud on this topic. The book is to be called "3264 and All That" and a draft was provided on the Harvard Math 232b web-page. For general introductions to algebraic geometry, I recommend "Algebraic Geometry" by Harris for an introduction and "The Geometry of Schemes" by Eisenbud and Harris for an introduction to the details.