CONSTRUCTING THE INTEGERS: $\mathbb{N}$, ORDINAL NUMBERS, AND TRANSFINITE ARITHMETIC

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10^{50} *is a long way from infinity.*
- Daniel Shanks

At PROMYS, we focus on number theory so we give an axiomatic description of $\mathbb{Z}$. In particular, $\mathbb{Z}$ is a commutative ring with identity with the following additional properties.

(1) There exists a non-empty $\mathbb{N} \subset \mathbb{Z}$ that is closed under addition and multiplication.
(2) For every $a \in \mathbb{Z}$, either $a = 0$, $a \in \mathbb{N}$, or $-a \in \mathbb{N}$.
(3) Every non-empty subset of $\mathbb{N}$ has a least element (with the definition $a > b$ means $a - b \in \mathbb{N}$).

However, it is also possible to construct the integers using set theory. However, it is no harder to construct the ordinal numbers, and as this will allow an exploration of transfinite arithmetic we will pursue the construction in more generality.

1. First Steps

If we are to construct the integers (and ordinal numbers), we want to do it with set theory and not rely on other number systems. Here we will begin describing the von Neumann construction for some small ordinals.

The simplest set is the empty set. It represents 0. Without introducing elements of number systems, there is only one other object that naturally arises: the set containing the empty set, denoted by $\{\emptyset\}$. This represents the number 1. There are two candidates for the number 2: either $\{\{\emptyset\}\}$ or $\{\emptyset, \{\emptyset\}\}$. We will use the latter, as it has two elements. Note that 2 is represented by the set containing 0 and 1. Thus the successor is a natural construction for ordinals. In general, we can represent natural numbers inductively: $n + 1$ is the set containing $0, 1, \ldots, n$ (represented using von Neumann's construction). Some are shown in Table 1.

Table 1. Some small ordinals

0	$\emptyset$
1	$\{\emptyset\}$
2	$\{\emptyset, \{\emptyset\}\}$
3	$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
4	$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$
5	$\{0, 1, 2, 3, 4\}$
6	$\ldots$

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Finally, a note about the name “ordinal”. In linguistics, these refer to the concepts first, second, third, etc. which are related to but distinct from the counting numbers one, two three, etc. The essential property is that they order things as opposed to counting them. The von Neumann construction captures this by having larger ordinals contain the smaller ones, so the ordering is very easy to see and define.

Definition 1. Let α and β be von Neumann ordinals. Then $\alpha < \beta$ if $\alpha \in \beta$.

Denote the smallest collection of ordinals containing $\emptyset$ and closed under the successor operation to be ω. There are subtleties in this definition which will be discussed in the next section. In the notation of PROMYS, ω would be $\mathbb{N}$.

We can then define addition and multiplication inductively on ω.

Definition 2. For $\alpha, \beta \in \omega$, either $\beta = 0$ or $\beta = \gamma + 1$ for $\gamma \in \omega$. In the first case, define $\alpha + \beta = \alpha + 0 = \alpha$ and $\alpha \cdot \beta = \alpha \cdot 0 = 0$. In the second, inductively define $\alpha + \beta = (\alpha + \gamma) + 1$ and $\alpha \cdot \beta = \alpha \cdot (\gamma + 1) = \alpha \cdot \gamma + \alpha$.

For example, as $2 = 1 + 1$ we can calculate that $2 + 2 = (2 + 1) + 1 = 3 + 1 = 4$, where the only property we are assuming about $1, 2, 3, 4$ is that 2 is the symbol with which we denote the successor of 1, 3 denotes the successor of 2, and so forth. Thus we prove the age-old fact that $2 + 2 = 4$.

With this definition, we can prove many of the basic arithmetic properties of the natural numbers. A few technical details are deferred to the next section, so the keen-eyed reader might notice some lacuna.

That 0 is the additive identity is clear.

Proposition 3. *If $\alpha, \beta, \gamma \in \omega$, then $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.*

Proof. If $\gamma = 0$, the assertion is that

$$(\alpha + \beta) + 0 = \alpha + (\beta + 0).$$

This follows from the definition of addition. Otherwise, let $\gamma = \delta + 1$, and assume the assertion holds for δ. Then repeatedly using the definition of addition and the inductive hypothesis

$$\begin{aligned}(\alpha + \beta) + \gamma &= (\alpha + \beta) + (\delta + 1) \\ &= ((\alpha + \beta) + \delta) + 1 \\ &= (\alpha + (\beta + \delta)) + 1 \\ &= \alpha + ((\beta + \delta) + 1) \\ &= \alpha + (\beta + (\delta + 1)) \\ &= \alpha + (\beta + \gamma).\end{aligned}$$

□

The commutative law requires a little more work.

Proposition 4. *If $\alpha, \beta \in \omega$, then $\alpha + \beta = \beta + \alpha$.*

Proof. First suppose $\beta = 0$. We wish to show that $\alpha + 0 = 0 + \alpha$. On one hand, $\alpha + \beta = \alpha + 0 = \alpha$. If $\alpha = 0$ as well, then $0 + \alpha = 0 + 0 = 0$. If $\alpha = \gamma + 1$ and the assertion holds for γ, then $0 + \alpha = 0 + (\gamma + 1) = (0 + \gamma) + 1 = \gamma + 1 = \alpha$. Thus $\alpha + 0 = 0 + \alpha$ for all $\alpha \in \omega$.

Now suppose $\beta = 1$. $\alpha + 1 = 1 + \alpha$ if $\alpha = 0$. Suppose $\alpha = \gamma + 1$ and the assertion holds for γ. Then using the associative law

$$\begin{aligned}\alpha + 1 &= (\gamma + 1) + 1 \\ &= (1 + \gamma) + 1 \\ &= 1 + (\gamma + 1) \\ &= 1 + \alpha.\end{aligned}$$

Finally suppose $\beta = \gamma + 1$ and $\alpha + \gamma = \gamma + \alpha$. Using the above calculation and the associative law gives

$$\begin{aligned}\alpha + \beta &= \alpha + (\gamma + 1) \\ &= (\alpha + \gamma) + 1 \\ &= (\gamma + \alpha) + 1 \\ &= \gamma + (\alpha + 1) \\ &= \gamma + (1 + \alpha) \\ &= (\gamma + 1) + \alpha \\ &= \beta + \alpha.\end{aligned}$$

□

Since the ordinals are the positive integers, there are of course no additive inverses. The remaining properties of ℕ can be dealt with similarly, but additional problems arise since we haven't actually defined what an ordinal number is.

2. Ordinal Numbers

This section will rigorously define what an ordinal number is. We will see there are many infinite ordinals in addition to the familiar ones in ω seen in the last section.

Ordinals are constructed out of well-ordered sets.

Definition 5. A relation $<$ on a set S is a strict total ordering if for $a, b, , c \in S$ $a \not< a$, $a < b$ and $b < c$ implies $a < c$, and exactly one of the following holds: $a < b$, $b < a$, or $a = b$.

Definition 6. A set is well-ordered with respect to an ordering if every non-empty subset has a least element.

For example, the integers are not well ordered but the natural numbers are.

Definition 7. A set S is transitive if $y \in X$ implies that $y \subset X$. Equivalently, if $z \in y$ and $y \in X$ then $z \in X$.

For example, the von Neumann ordinals seen in the last section are transitive.

Definition 8. An ordinal is a transitive set well-ordered under the relation $x < y$ if $x \in y$.

Definition 9. Let α and β be ordinals. We write $\alpha < \beta$ if $\alpha \in \beta$.

Note that if both α and β were contained in a larger ordinal γ, this matches the ordering on γ. In particular, the collection of all ordinals is well-ordered.

Example 10. The empty set and $\{\emptyset\}$ are ordinals for trivial reasons. The set $T = \{\emptyset, \{\emptyset\}\}$ is an ordinal, as $\emptyset \in \{\emptyset\}$ and $\{\emptyset\} \in T$ while $\emptyset \in T$.

There are two main ways to construct new ordinals: taking the successor of an ordinal and picking an element of an ordinal.

Definition 11. Let α be an ordinal, and define the successor of α, denoted by $\alpha + 1$, to be $\{\alpha\} \cup \alpha$.

Lemma 12. *$\alpha + 1$ is an ordinal.*

Proof. This proof will be spelled out in complete detail. Note that α is the largest element of $\alpha + 1$, and we will show $\in$ is still a strict ordering. Pick $x, y, z \in \alpha + 1$. If $x \in \alpha$, $x \notin x$ as α is an ordinal, while if $x = \alpha$ then $x \notin x$ as α is strictly ordered. If $x \in y$ and $y \in z$, if $z \neq \alpha$ then $x < z$ as $\in$ is transitive on α. If $z = \alpha$, then $x \in \alpha$. If $x, y \in \alpha$ then $x < y$, $y < x$, or $x = y$. If $x \neq \alpha$ but $y = \alpha$, then $x \in \alpha$ so $x < y$. If $y = \alpha \in x$, then as α is transitive $\alpha \in \alpha$. But $\in$ is a strict ordering on α, so $y \notin x$. Finally if $x = y = \alpha$ then $x \notin y$ by the same reasoning. Therefore $\in$ is a strict ordering.

$\alpha + 1$ is transitive, for if $y \in \alpha + 1$ then either $y = \alpha$ and $\alpha \subset \alpha + 1$, or $y \in \alpha$. But α is transitive, so $y \subset \alpha \subset \alpha + 1$. It is well ordered: note that the only new element, α, is greater than any $x \in \alpha$. Thus a non-empty subset of $\alpha + 1$ either contains only α, in which case it has a least element, or the intersection with α is non-empty and hence has a least element as α is well-ordered. $\square$

Most ordinals are successors. The ones that are not are quite interesting.

Definition 13. A limit ordinal is an ordinal α such that there does not exist an ordinal β with $\beta + 1 = \alpha$.

We will soon see that ω is a limit ordinal. The second way to construct ordinals is as elements of larger ordinals.

Lemma 14. *Let α be an ordinal, and $\beta \in \alpha$. Then β is an ordinal.*

Proof. Since α is transitive, $\beta \subset \alpha$. Therefore β is well-ordered by $\in$ since α is. Now suppose $\gamma \in \beta$ and $\delta \in \gamma$. Since $\in$ is a strict total order on α, $\delta \in \beta$. Therefore β is transitive. $\square$

Similarly, we can identify an initial segment of an ordinal as an ordinal.

Lemma 15. *Let A be an initial segment of an ordinal α (this means it is a subset with the property that for $x \in A$ and $y \in \alpha$, $y < x$ implies $y \in A$). Then A is an ordinal, and either $A \in \alpha$ or $A = \alpha$.*

Proof. First, we show that A is an ordinal. If $x \in y$ and $y \in z$ for $x, y, z \in A$, then $x \in z$ as $A \subset \alpha$ is ordered. A is well-ordered as any subset is also a subset of α which is well-ordered. A is transitive, for if $y \in A$ as A is initial any x with $x \in y$ also satisfies $x \in A$. Thus $y \subset A$.

Now suppose $A \neq \alpha$. For $\beta \in \alpha$, either $\beta \in A$ or $\beta > \gamma$ for every $\gamma \in A$. Thus there exists a β such that $A \subset \beta$ as β contains all smaller ordinals. Pick the least such β, and suppose $A \neq \beta$. Then there is a $\delta \in \beta$ with $A \subset \delta$. But α is transitive, so $\delta \in \alpha$. Thus β is not minimal. Hence $A = \beta$, so $A \in \alpha$. $\square$

This construction gives a proof of the law of trichotomy.

Theorem 16 (Trichotomy)**.** *If α and β are ordinals, either $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$.*

Proof. Given distinct α and β, let $A = \alpha \cap \beta$. A is an initial segment of α since if $x \in \alpha$ and if $x < y \in \alpha$ then $x \in \alpha$. Likewise for β. By the lemma, either $A \in \alpha$ or $A = \alpha$. Likewise, either $A \in \beta$ or $A = \beta$. If $A \in \alpha$ and $B \in \beta$, then $A \in A$. However this contradicts the fact that $\in$ is a strict order. Therefore $\alpha < \beta$, $\beta < \alpha$, or $\alpha = \beta$. □

Although this proof is fairly short, a surprising amount of technical machinery went into proving it. Although it is possible to prove the basic properties of $\mathbb{Z}$ using ordinals, the effort involved is large while the properties are simple.

We will now show that the set ω defined in the last section is an ordinal.

Definition 17. Let ω be the smallest set of ordinals closed under successor such that $\emptyset \in \omega$.

Our experience shows $\omega = \{0, 1, 2, \ldots\}$.

Remark 18. This is not a very good definition from the point of view of axiomatic set theory. We also cannot talk about finite sets without understanding ordinals. Instead, we should define a finite ordinal x to be an ordinal satisfying the following three properties:

- If $z \in y$ and $y \in x$, then $z \in x$.
- Each nonempty x has exactly one limit element (an element that is not the successor of anything).
- Each nonzero x has exactly one element with no successor, which we call $x - 1$. x satisfies $x = (x - 1) \cup \{x - 1\}$.

The axiom of infinity then implies that the set of finite ordinals, ω, is a set. [3, Section 2.3]. This allows a rigorous definition of what it means for a set to be finite: there is a bijection to an element of ω.

We now focus on our first example of an infinite ordinal.

Proposition 19. *ω is a limit ordinal.*

Proof. As ω is a set of ordinals, it is strictly well-ordered by $\in$. We need to show that it is transitive. If not, let $A = \{\alpha \in \omega : \alpha \not\subset \omega\}$. Let its least element be α^*. Then $\alpha^* \neq \emptyset$. Suppose $\alpha^* = \beta + 1$ for some $\beta \in \omega$. Then $\beta \subset \omega$, and $\alpha^* = \beta \cup \{\beta\} \subset \omega$, which is a contradiction. Therefore α^* is not a successor of anything in ω. But ω is the smallest set containing $\emptyset$ and closed under successor.

Suppose ω is a successor, so $\alpha + 1 = \omega$. Then $\alpha < \omega$, so $\alpha \in \omega$. Since ω is closed under taking successor, $\omega \in \omega$. But $\in$ is a strict ordering on any ordinal. Thus ω is a limit ordinal. □

In the next section, we will construct various other countable ordinals in terms of ω using the ordinal arithmetic. For now, one last definition.

Definition 20. Let ω_1 denote the first uncountable ordinal.

Remark 21. The collection of all ordinals is not a set. If it were, then it would be transitive and well-ordered by $\in$, so it would be a member of itself. This is a contradiction.

3. Ordinal Arithmetic

We will define addition inductively. A formal statement of induction is as follows:

Theorem 22 (Induction). *Let ϕ be a logical formula, and suppose that we know that for all ordinals β, if for all $\gamma < \beta$, $\phi(\gamma)$ is true then $\phi(\beta)$ is true. Then ϕ is true for every ordinal.*

Proof. Note that taking $\beta = \emptyset$ means that $\phi(0)$ is true. In general, let S be the collection of ordinals α for which $\phi(\alpha)$ is false. Assume it is non-empty. Since the ordinals are well-ordered, there is a least element γ. Then $\phi(\beta)$ is true for all $\beta < \gamma$, which implies $\phi(\gamma)$ is true by the inductive hypothesis. Thus S is empty. □

Definition 23 (Addition)**.** For all ordinals α, β, let $\alpha + 0 = \alpha$, $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, and if β is a non-zero limit let $\alpha + \beta = \cup_{\gamma<\beta} \alpha + \gamma$.

This agrees with the definition of addition given for ω earlier as 0 is the only limit ordinal in ω. However, there are many other limit ordinals such as ω.

Definition 24 (Multiplication)**.** For all ordinals α and β, define their product as follows. If $\beta = \gamma + 1$, inductively define $\alpha \cdot \beta = \alpha\gamma + \alpha$. If β is a limit ordinal, define $\alpha \cdot \beta = \cup_{\gamma<\beta} \alpha \cdot \gamma$.

Let us compute some examples.

Example 25. ω is the smallest infinite ordinal. $\omega + 1$ is the successor of ω. It is not equal to ω as $\omega \in \omega + 1$ but $\omega \notin \omega$. The same reasoning shows that $\omega + 1, \omega + 2, \omega + 3, \ldots$ are all distinct.

However, $1 + \omega = \omega$. As ω is a limit ordinal, $1 + \omega = \cup_{n\in\omega}(1 + n)$. But $1 \cup 2 \cup 3 \ldots = \omega$. Therefore addition of ordinals is not commutative in general.

As an example of multiplication, $2 \cdot \omega = \cup_{n\in\omega} 2n = \omega$. On the other hand,

$$\omega \cdot 2 = \omega + \omega = \cup_{n\in\omega}(\omega + n)$$

is strictly larger than $\omega + n$ for any $n \in \omega$.

Note that these operations are compatible with the ordering. For example, the following follows from the definition of addition and induction.

Lemma 26. *Let α, β and γ be ordinals with $\beta < \gamma$. Then $\alpha + \beta < \alpha + \gamma$.*

We can now give a more general version of the associative law.

Proposition 27 (Associativity)**.** *Let α, β, γ be ordinals. Then $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.*

First a lemma.

Lemma 28. *Let α be an ordinal and A be a set of ordinals.* $\sup_{\beta\in A}(\alpha+\beta) = \alpha+\sup_{\beta\in A}(\beta)$.

Proof. Let γ be an ordinal with $\gamma < \sup_{\beta\in A}(\alpha + \beta)$. Then $\gamma < \alpha + \beta$ for some $\beta \in A$. This means $\gamma < \alpha + \sup_{\beta\in A} \beta$, for $\gamma < \alpha + \beta < \alpha + \sup_{\beta\in A} \beta$ by Lemma 26. Therefore the ordinals less than $\sup_{\beta\in A}(\alpha + \beta)$ and the ordinals less than $\alpha + \sup_{\beta\in A} \beta$ are the same. Since set inclusion is the ordering, the two are equal. □

We can now prove associativity.

Proof. We proceed by induction. If γ is a successor, the proof given in the previous section for finite ordinals works verbatim. Otherwise, γ is a limit ordinal. Then by the lemma, inductive hypothesis, and definition of addition

$$\begin{aligned}(\alpha + \beta) + \gamma &= \sup_{\delta<\gamma}(\alpha + \beta) + \delta \\ &= \sup_{\delta<\gamma} \alpha + (\beta + \delta) \\ &= \alpha + \sup_{\delta<\gamma}(\beta + \delta) \\ &= \alpha + (\beta + \gamma).\end{aligned}$$

□

In a similar manner, we can prove the distributive law and associativity for multiplication.

Theorem 29. *Let α, β, and γ be ordinals. Then $(\alpha\cdot\beta)\cdot\gamma = \alpha\cdot(\beta\cdot\gamma)$ and $\alpha(\beta+\gamma) = \alpha\cdot\beta+\alpha\cdot\gamma$.*

Proof. We will prove the distributive law as the prove of associativity is similar to the associativity proof for addition. Proceed by induction on γ. If $\gamma = 0$, the result is just the definitions. If $\gamma = \delta+1$ and the theorem holds for δ, then the definition of multiplication says that

$$\alpha(\beta+\gamma) = \alpha((\beta+\delta)+1) = \alpha(\beta+\delta)+\alpha.$$

By the inductive hypothesis,

$$\alpha(\beta+\gamma) = \alpha\beta+\alpha\delta+\alpha$$

The definition of addition then shows this is $\alpha\beta+\alpha\gamma$.

Otherwise, γ is a limit ordinal and we may assume the result holds for all smaller ordinals.

$$\begin{aligned}\alpha(\beta+\gamma) &= \alpha \sup_{\delta<\gamma}(\beta+\delta)\\ &= \sup_{\delta<\gamma}(\alpha(\beta+\delta))\\ &= \sup_{\delta<\gamma}(\alpha\beta+\alpha\delta)\\ &= \alpha\cdot\beta+\alpha\sup_{\delta<\gamma}\delta\\ &= \alpha\cdot\beta+\alpha\cdot\gamma.\end{aligned}$$

We use the definitions of the operations, the inductive hypothesis, and an analogue of Lemma 28. □

As you can see, while it is possible to prove the ring axioms using these definitions, it is quite involved. There is a good reason why we use an axiomatic characterization of $\mathbb{Z}$ at PROMYS. Furthermore, note that there is no analogue of the commutative law. We saw an example of this as $2\omega = \omega$ but $\omega\cdot 2 = \omega+\omega > \omega$. Likewise with addition, $1+\omega = \omega$. Of course, when we restrict to finite ordinals there are analogues as discussed in Section 1.

We will close with an explanation of the mysterious ordinal $\omega\cdot\omega$. By definition, $\omega\cdot\omega = \cup_{\gamma\in\omega}\omega\cdot\gamma$. So we can think of it as an infinite number of copies of ω ordered one after another. In fact, we can find a picture of it inside the real numbers.

Theorem 30. *Let $x_{n,m}$ be the real number with decimal expansion $.1\ldots101\ldots10000\ldots$ where there are n 1s in the first block of 1s and m in the second. The set $\{x_{n,m} : n,m\in\omega\}$ is a subset of $\mathbb{R}$. The ordering they inherit from $\mathbb{R}$ agrees with the ordering of $\omega\cdot\omega$.*

Proof. An element of $\omega\cdot\omega$ can be thought of as a pair (n,m) where $n,m\in\omega$ and n counts which copy of ω the element is in and m represents where the element is in that copy of ω. So $(3,2)$ is the ordinal $\omega\cdot 3+2\in\omega\cdot\omega$. $(n,m) < (n',m')$ if $n < n'$ or if $n = n'$ and $m < m'$. On the other hand, $x_{n,m} < x_{n',m'}$ if and only if $n < n'$ (so the first 0 occurs first in $x_{n,m}$) or if $n = n'$ and $m < m'$ (so the second string of 1s ends first in $x_{n,m}$). □

Similar things can be done with $\omega\cdot\omega\cdot\omega$ and even larger powers (or exponents). However, because any subset of $\mathbb{R}$ that is well-ordered with respect to the induced ordering must be countable (Proof: look at open intervals between successive elements: each must contain a rational), ω_1 is so big it cannot appear inside $\mathbb{R}$ at all. To illustrate this, show as an exercise that

$$\omega+\omega_1 = \omega_1.$$

4. Constructing $\mathbb{Z}$

Taking the finite ordinals gives a model for $\mathbb{N} \cup \{0\}$. We can prove all of the standard properties of $\mathbb{N}$. How do we introduce the negative numbers? It turns out this is very easy compared to what we have already done.

Definition 31. Define the following equivalence relation on pairs $(a, b) \in \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}$: $(a, b) \simeq (c, d)$ if and only if $a + d = b + c$. Define $\mathbb{Z}$ to be $\mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}$ modulo this equivalence relation. $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \cdot (c, d) = (ac + bd, ad + bc)$. Let $\mathbb{N}$ be the equivalence classes corresponding to $(a, 0)$ for $a \in \omega$ and $a \neq 0 = \emptyset$.

The pair (a, b) represents $a - b$. Note that $(a - b)(c - d) = ac + bd - bc - bd$ which explains the mysterious definition of multiplication. It is quite straightforward to verify the following:

Theorem 32. *The set $\mathbb{Z}$ constructed above satisfies the properties of $\mathbb{Z}$ listed in the introduction.*

Proof. All of the assertions above about $\mathbb{N}$ follow from the theory of finite ordinals. The operations being well-defined and the arithmetic assertions about $\mathbb{Z}$ are trivial consequence of the arithmetic of ordinals. For example, $(a, b) + (c, d) = (a + c, b + d) = (c + a, d + b) = (c, d) + (a, b)$ since addition of finite ordinals is commutative. For the law of trichotomy, we know that either $a > b$, $a = b$, or $a < b$ for ordinals. This implies the law of trichotomy for $\mathbb{N} \subset \mathbb{Z}$. $\square$

Given $\mathbb{Z}$, one can construct $\mathbb{Q}$ by looking at fractions, $\mathbb{R}$ by completing or looking at Dedekind cuts, and $\mathbb{C}$ by adjoining i. So all of analysis and number theory can be reduced to set theory!

5. References

To learn more about ordinals and these constructions, you need a good understanding of axiomatic set theory. The main reference for this talk was An Introduction to Modern Set Theory by Roitman [3]. Naive Set Theory by Halmos is a gentle introduction [1]. Jech's Set Theory is much more comprehensive and advanced [2]. A good understanding of the size of infinite sets is helpful as well, and allows cardinals and cardinal arithmetic to be defined. Finally, this parting fact: additional axioms are necessary to show the existence of large cardinals (large in a precise sense). However, their existence shows that the axioms of $\mathbb{Z}$ are consistent, meaning no contradictions can be derived!

References

1. Paul Halmos, *Naive set theory*, D. van Nostrand, 1960.
2. T.J. Jech, *Set theory*, Academic Press, 1978.
3. Judith Roitman, *An introduction to modern set theory*, Wiley-Interscience, 1990.