REPRESENTATIONS OF THE SYMMETRIC GROUP VIA YOUNG TABLEAUX

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As a concrete example of the representation theory we have been learning, let us look at the symmetric groups S_n and attempt to understand their representations. We previously calculated the character table of S_4.

	1	6	8	6	3
	(1)	(1 2)	(1 2 3)	(1 2 3 4)	(1 2)(3 4)
trivial	1	1	1	1	1
sign	1	-1	1	-1	1
standard	3	1	0	-1	-1
sign $\otimes$ standard	3	-1	0	1	-1
other	2	0	-1	0	2

Table 1. Irreducible Characters of S_4

The trivial, sign, and standard representations are natural representations, and the tensor product of the sign and standard representation is a natural operation to do. We found the last irreducible representation by using the orthogonality relations, and later constructed the representation by looking at a quotient of S_4.

Here is a partial character table of S_5. It is easy to find the conjugacy classes, as they correspond to the 7 partitions of 5.

	1	10	20	30	24	15	20
	(1)	(1 2)	(1 2 3)	(1 2 3 4)	(1 2 3 4 5)	(1 2)(3 4)	(1 2)(3 4 5)
trivial	1	1	1	1	1	1	1
sign	1	-1	1	-1	1	1	-1
standard	4	2	1	0	-1	0	-1
sign $\otimes$ standard	4	-2	1	0	-1	0	1

Table 2. Some Irreducible Characters of S_5

These four representations are the obvious ones. There are three more, so the orthogonality relations don't help. We can search for more in an ad hoc way (and find, for example, that $\Lambda^2(V)$, where V is the standard representation is irreducible). However, we want a more principled way to proceed. The answer is the theory of Young Tableaux. An algebraic prospective is presented in Fulton and Harris, Lecture 4 [1]. There is a combinatorial focus in Sagan [2]. Zhao presents a readable survey without most of the proofs [3] that is freely available.

1. Young Tableaux

Given a partition of n, represent it by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, where $n = \lambda_1 + \lambda_2 + \ldots + \lambda_r$. By convention, we order the partition so $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$.

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Definition 1. Given a partition λ of n, a Young diagram of shape λ is an array of boxes arranged in rows. There are λ_i boxes in row i, the boxes are left justified, and by the convention on partitions the lengths of rows are non-increasing.

$$\ydiagram{2,1,1,1} \qquad\qquad \begin{ytableau} 1 & 2 \\ 4 \\ 5 \\ 3 \end{ytableau}$$

FIGURE 1. A Young diagram and Young tableau of shape $(2,1,1,1)$

Definition 2. A Young tableau of shape λ is an assignment of the numbers $1, 2, \ldots, n$ to the n boxes of the Young diagram associated to λ.

The symmetric group S_n acts on Young tableau by acting on the entries. For example,

$$(1\,3)\begin{ytableau} 1 & 2 \\ 4 \\ 5 \\ 3 \end{ytableau} = \begin{ytableau} 3 & 2 \\ 4 \\ 5 \\ 1 \end{ytableau}.$$

The action of S_n on the set of all tableau isn't actually interesting, since it is simply the regular representation where we write numbers 1 through n in a pretty picture. Instead, we need to act on tabloids.

Definition 3. A Young tabloid is an equivalence class of Young tableau under the relation that two tableau are equivalent if each row contains the same elements.

Despite being equivalence classes, we will continue to denote tabloids by simply drawing a representative tableau.

Definition 4. Let M^λ be the representation of S_n whose basis is indexed by the set of Young tabloids and the action on the basis is the action on the tabloids.

This is an example of a permutation module. For example, if $\lambda = (3,2)$ then we have the following action:

$$(1\,2\,3)\left[2\cdot\begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \end{ytableau} - \begin{ytableau} 1 & 2 & 4 \\ 5 & 3 \end{ytableau}\right] = 2\cdot\begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \end{ytableau} - \begin{ytableau} 2 & 3 & 4 \\ 1 & 5 \end{ytableau}$$

Example 5. Let $\lambda = (5)$. The permutation module M^λ is the trivial module, as all tableaux of this shape have just one row, hence all are equivalent. If $\lambda = (1,1,1,1,1)$, we get the regular representation of S_5. There is only one element in each row, so there is one basis vector for each assignment of $\{1,2,3,4,5\}$ to the 5 boxes.

If $\lambda = (4,1)$, there are five tabloids:

$$\begin{ytableau} 1 & 2 & 3 & 4 \\ 5 \end{ytableau}, \quad \begin{ytableau} 1 & 2 & 3 & 5 \\ 4 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 4 & 5 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 4 & 5 \\ 2 \end{ytableau} \quad \begin{ytableau} 2 & 3 & 4 & 5 \\ 1 \end{ytableau}.$$

What is in the box in the second row uniquely determines the class. A permutation $\sigma \in S_5$ acts on these by permuting the element in the second row, so M^λ is the permutation representation. Recall it decomposes as a direct sum of the trivial and standard representations.

Now that we have a supply of new representations of S_n, we need to analyze them. The dimension is very easy to calculate.

Proposition 6. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of n. Then*

$$\dim M^\lambda = \frac{n!}{\lambda_1!\lambda_2!\ldots\lambda_r!}.$$

Proof. A basis is given by tabloids of shape λ. There are $n!$ ways to assign the numbers $\{1, 2, \ldots, n\}$ to the boxes of a Young diagram of shape λ. Any permutation that fixes the rows preserves the equivalence classes of Young tabloids. There are $\lambda_1!$ permutations that permute the first row, $\lambda_2!$ the second, and so forth. Thus there are $\frac{n!}{\lambda_1!\lambda_2!\ldots\lambda_r!}$ tabloids. □

The next task is to evaluate the character of M^λ. It is not too hard to do this by hand in any particular case. There is also a nice general formula.

Proposition 7. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of n and $g \in S_n$. Let $(m_1, \ldots, m_r)$ be the cycle shape of g (this means g is a product of a m_1 cycle, a m_2 cycle, and so forth). The character of the representation of S_n on M^λ, evaluated at g, is equal to the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_l^{\lambda_l}$ in the product*

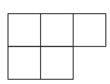
$$\prod_{i=1}^{r} \left(x_1^{m_i} + x_2^{m_i} + \ldots + x_l^{m_i} \right).$$

Proof. In any permutation representation, the number of fixed points of the action is the trace (simply write down the action as a matrix and look on the diagonal). Which tabloids will g fix? It fixes exactly those tabloids for which each cycle of g permutes the elements of a single row. In other words, if we pick an assignment of the m_i to rows such that $\sum_{j=1}^{t} m_{i,j} = \lambda_i$, then the tabloid whose ith row consists of the λ_i elements occurring in these cycles will be fixed. The generating function

$$\prod_{i=1}^{r} \left(x_1^{m_i} + x_2^{m_i} + \ldots + x_l^{m_i} \right)$$

represents the ways to assign the cycles to rows. When expanding, the exponent of x_i is the number of elements assigned to row i. To get λ_i elements in each row, we are looking at the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_l^{\lambda_l}$. This is the number of tabloids fixed by g, and hence the trace. □

Example 8. Consider the partition $\lambda = (3, 2)$. We have

Young diagram:  $\dim M^\lambda = \frac{5!}{3!2!} = 10$ and $\chi_\lambda((1\,2\,3)) = 2$.

For the last, note that the permutation $(1\,2\,3)$ is actually one three cycle and two one cycles, so we want the coefficient of $x_1^3 x_2^2$ in

$$(x_1^3 + x_2^3)(x_1 + x_2)^2.$$

It is very easy to compute the rest of the character of M^λ using the same method.

	(1)	(1 2)	(1 2 3)	(1 2 3 4)	(1 2 3 4 5)	(1 2)(3 4)	(1 2)(3 4 5)
$\chi_{(3,2)}$	10	4	1	0	0	2	1

Now that we have this new representation, we can try to decompose it and find additional irreducible representations. The inner product of this character with itself is

$$\frac{1}{120}(10^2 + 4^2 \cdot 10 + 20 \cdot 1 + 2^2 \cdot 15 + 20) = 3.$$

This means there are three irreducible representations, as the only way 3 can be written as a sum of squares is as $1^2 + 1^2 + 1^2$. A calculation shows that

$$(\chi_{(3,2)}, \chi_{triv}) = 1 \quad \text{and} \quad (\chi_{3,2}, \chi_{standard}) = 1.$$

Subtracting these two characters out gives another irreducible character, call it χ. The tensor product with the sign representation is again an irreducible character, so we now know 6 rows of the character table. The orthogonality relations are now enough to fill in the rest.

	1	10	20	30	24	15	20
	(1)	(1 2)	(1 2 3)	(1 2 3 4)	(1 2 3 4 5)	(1 2)(3 4)	(1 2)(3 4 5)
trivial	1	1	1	1	1	1	1
sign	1	-1	1	-1	1	1	-1
standard	4	2	1	0	-1	0	-1
sign $\otimes$ standard	4	-2	1	0	-1	0	1
χ	5	1	-1	-1	0	1	1
$\chi\otimes$ sign	5	-1	-1	1	0	1	-1
other	6	0	0	0	1	-2	0

TABLE 3. The Character Table of S_5

It seems we were lucky that M^λ contained only one new irreducible representation. In fact, it turns out is possible to partially order the partitions so that M^λ contains only permutations greater than or equal to λ (see Corollary 19). So this suffices to compute the character table. However, it would be nicer to have a concrete description of the irreducible modules, and a better method to calculate their characters.

2. Specht Modules

We will construct Specht Modules to better understand irreducible representations of S_n.

Definition 9. Let t be a Young tableau of shape λ, and C_t be the permutations of S_n which preserve the columns of t. Define

$$e_t = \sum_{\pi \in C_t} \operatorname{sign}(\pi)\pi(t) \in M^\lambda.$$

Let the Specht module S^λ be the span of the e_t where t is a tableau of shape λ.

For example, if $t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & 4 \\ \cline{1-2}\end{array}$ then

$$e_t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & 4 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 5 & 2 & 3 \\ \hline 1 & 4 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 1 & 4 & 3 \\ \hline 5 & 2 \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 5 & 3 & 4 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}.$$

Remember these are tabloids in M^λ, so the order of the rows doesn't matter.

Lemma 10. *S^λ is a representation of S_n.*

Proof. To show it is a representation, we need to show that S^λ is actually a sub-module of M^λ. In particular, we need to check that $\pi(e_t) \in S^\lambda$ for t a tableau of shape λ. However,

$$\begin{aligned} e_{\pi t} &= \sum_{\sigma \in C_{\pi t}} \operatorname{sign}(\sigma)\sigma(\pi(t)) = \sum_{\sigma \in \pi C_t \pi^{-1}} \operatorname{sign}(\sigma)\sigma(\pi(t)) \\ &= \sum_{\sigma' \in C_t} \operatorname{sign}(\pi\sigma'\pi^{-1})\pi\sigma'\pi^{-1}(\pi t) \\ &= \pi \sum_{\sigma' \in C_t} \operatorname{sign}(\sigma')\sigma'(t) = \pi e_t. \end{aligned}$$

□

Example 11. In general, it requires some thought to find a basis for S^λ. However, in a few cases it is quite easy.

Consider $t = \begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline\end{array}$. S^t is the trivial representation, because all tableaux of this shape are row equivalent.

Consider $\lambda = (1, 1, 1, 1, 1)$. S^λ is the sign representation, because all permutations fix the columns of any tableau of this shape. Fix a tableau t of shape λ. Any other tableau is of the form πt for $\pi \in S_n$. On the other hand, any $\sigma \in S_n$ preserves the columns of t. Thus

$$\pi e_t = e_{\pi t} = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\sigma\pi(t) = \operatorname{sign}(\pi)e_t.$$

Therefore S^λ is the one dimensional sign representation.

Finally consider $\lambda = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$. S^λ is the standard representation. To see this, let f_i denote the tabloid with i in the bottom box. Consider a tableau

$$t = \begin{array}{|c|c|c|c|}\hline a & i & j & k \\ \hline b \\ \cline{1-1}\end{array}.$$

A direct calculation shows that $e_t = \begin{array}{|c|c|c|c|}\hline a & i & j & k \\ \hline b \\ \cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline b & i & j & k \\ \hline a \\ \cline{1-1}\end{array}$. Therefore S^λ is the span of all such elements, so it is

$$S^\lambda = \{c_1 f_1 + \ldots + c_n f_n : c_1 + c_2 + c_3 + \ldots + c_n = 0\}.$$

Our goal is to prove the following result.

Theorem 12. *The Specht modules S^λ, where λ is a partition of n, form a complete list of irreducible complex representations of S_n.*

The proof will proceed in two steps. First we show that S^λ is an irreducible sub-module of M^λ, second we show that $S^\lambda \neq S^\mu$ if $\mu \neq \lambda$. Since the number of conjugacy classes of S_n is the number of partitions is the number of non-isomorphic Specht modules, we have a complete list of irreducibles.

2.1. **Proof that Specht modules are irreducible.** There are three main ingredients in the proof. The first is putting an inner product on M^λ. Since we specified a basis for M^λ, the set of tabloids, we pick the usual dot product as the inner product. In particular, $(t, s) = \delta_{t,s}$, so the inner product is non-zero if and only if the tableaux s and t are row equivalent.

The second idea is a "projection" operation. For a tableau t, define $\kappa_t \in \mathbb{C}S^n$ as

$$\kappa_t = \sum_{\pi \in C_t} \operatorname{sign}(\pi)\pi.$$

Note that if it acts on the tabloid t we recover e_t.

The third ingredient is the sub-module lemma.

Lemma 13 (Submodule Lemma)**.** *Let U be a sub-module of M^λ. Then either*

$$S^\lambda \subset U \quad \text{or} \quad U \subset (S^\lambda)^\perp.$$

Proof. Let t be a tableau of shape λ, and let $u \in U$. Write $u = \sum c_i t_i$ where the t_i are tabloids of shape λ. We have

$$\begin{aligned}\kappa_t u = \kappa_t(\sum_i c_i t_i) &= \sum_{\pi \in C_t} \sum_i \operatorname{sign}(\pi) c_i \pi(t_i) \\ &= \sum_i \sum_{\pi \in C_t} \operatorname{sign}(\pi) c_i \pi(t_i) \\ &= \sum_i c_i e_{t_i}.\end{aligned}$$

However, I claim that $e_{t_i} = \pm e_t$. Because all the terms in the sum defining e_t are tabloids, we may freely permute the rows of t without changing the value of e_t. Pick a permutation π which takes the t to a tableau which is row equivalent to t_i. Then modify π so that it only permutes elements within the columns of t: this is possible as we only care about which row a given number ends up in and t and t_i have the same shape. Thus $t_i = \pi t$ as tabloids, so $e_{t_i} = \pi e_t$. Therefore $\kappa_t u$ is a multiple of e_t. (This is why it is called a projection.)

Suppose $\kappa_t u$ is a non-zero multiple of e_t. Then $e_t \in U$. Since S^λ is a cyclic module generated by e_t (see Lemma 10), $S^\lambda \subset U$.

Otherwise, suppose for all choices of $u \in U$ and tableau t, $\kappa_t u = 0$. Then

$$(u, e_t) = (u, \sum_{\pi \in C_t} \operatorname{sign}(\pi) \pi t) = \sum_{\pi \in C_t} \operatorname{sign}(\pi)(u, \pi t).$$

The inner product satisfies $(u, \pi t) = (\pi^{-1} u, t)$ as $u = \pi t$ iff $\pi^{-1} u = t$. Also, note $\operatorname{sign}(\pi) = \operatorname{sign}(\pi^{-1})$. Thus

$$(u, e_t) = (\sum_{\pi \in C_t} \operatorname{sign}(\pi^{-1}) \pi^{-1} u, t) = (\kappa_t u, t) = 0$$

Therefore $U \subset (S^\lambda)^\perp$. ☐

Corollary 14. *S^λ is an irreducible complex representation of S_n.*

Proof. Let U be a sub-representation of S^λ. By the sub-module lemma, either $S^\lambda \subset U$ or $U \subset (S^\lambda)^\perp$. In characteristic 0, $(v, v) = 0$ implies $v = 0$, so in the second case $U = 0$. In the first, $U = S^\lambda$. ☐

2.2. **Proof that Specht modules are distinct.** The main ingredient in this proof is an ordering on Young diagrams.

Definition 15. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$. λ dominates μ, written $\lambda \geq \mu$, if for each $1 \leq i \leq \max(r, s)$ we have

$$\lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i.$$

By convention, if $r > s$, we set $\mu_i = 0$ for $i > s$, and vice versa.

Informally, $\lambda \geq \mu$ if λ is short and fat and μ is tall and skinny. For example,

$$\ydiagram{3,2} \geq \ydiagram{3,1,1}\,.$$

This is a partial ordering on the set of partitions of n (Young diagrams). It is only a total ordering for a few small values of n.

Lemma 16 (Dominance Lemma)**.** *Let t and s be tableau of shape λ and μ. If, for each i, the elements of row i of s are all in different columns of t, then $\lambda \geq \mu$*

Proof. Sort the entries of each column of t so that elements in the first i rows of s occur in the first i rows of t. This is possible as the elements of row i of t are in different columns of s. This implies that

$$\lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i.$$

Thus $\lambda \geq \mu$. ☐

Theorem 17. *Let $\theta \in \mathrm{Hom}(S^\lambda, M^\mu)$ be a nonzero map of representations. Then $\lambda \geq \mu$.*

Proof. Pick a vector e_t on which θ is non-zero. Now

$$0 \neq \theta(e_t) = \theta(\kappa_t t) = k_t(\sum_i c_i s_i)$$

where the s_i are tabloids in M^μ. Suppose $k_t s \neq 0$ for some s appearing in this sum. If there were numbers b and c in the same row of s and same column of t, then pick a set of coset representatives for $C_t/(b\,c)$, say $\sigma_1 \ldots, \sigma_r$. Then $\sigma_1, \ldots, \sigma_r, (b\,c)\sigma_1, \ldots, (b\,c)\sigma_r$ are the elements of C_t. However, σ_i and $(b\,c)\sigma_i$ have the opposite signs but act the same way on the tabloid s as b and c are in the same row. Applying κ_t, they cancel out. Therefore $\kappa_t s = 0$. Therefore there are no elements b and c in the same row of s but in the same column of t. By the Dominance Lemma $\lambda \geq \mu$ ☐

Corollary 18. *If $S^\mu \simeq S^\lambda$, then $\mu = \lambda$.*

Proof. We can include $S^\mu \hookrightarrow M^\mu$ and $S^\lambda \hookrightarrow M^\lambda$. As there is a nonzero map in both directions, the theorem implies $\mu \geq \lambda$ and $\lambda \geq \mu$. ☐

Corollary 19. *The only irreducible representations in M^μ are S^λ for $\lambda \geq \mu$.*

In particular, this implies that we can induct on the poset to find irreducible representations of S_n inside M^μ: as long as we have already found S^λ for $\lambda \geq \mu$ via induction (by looking in M^λ, say) the only other irreducible occurring in M^μ is S^μ.

Remark 20. In fact, there is an explicit formula (Young's Rule) for the decomposition of M^μ into irreducibles in terms of the Kostka numbers. For more information, see the section on Kostka numbers and Young's Rule [2, Section 2.11].

3. Combinatorics of Specht Modules

Just as the dimension and character of M^λ could be computed combinatorially, there are combinatorial methods to compute these for S^λ. Proofs can be found in Chapter 2 and 3 of Sagan's book [2]. Many of these are similar to the (easier) statements for M^λ.

Definition 21. A standard Young tableau is a Young Tableau such that every row and every column is increasing.

Proposition 22. *The set $\{e_t\}$ where t is a standard Young tableau of shape λ form a basis for S^λ.*

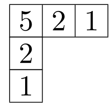
For example, there are six standard Young tableau with $\lambda = (3, 1, 1)$:

$$\begin{array}{ccc}1&2&3\\4&&\\5&&\end{array}\quad\begin{array}{ccc}1&2&4\\3&&\\5&&\end{array}\quad\begin{array}{ccc}1&2&5\\3&&\\4&&\end{array}\quad\begin{array}{ccc}1&3&4\\2&&\\5&&\end{array}\quad\begin{array}{ccc}1&3&5\\2&&\\4&&\end{array}\quad\text{and}\quad\begin{array}{ccc}1&4&5\\2&&\\3&&\end{array}.$$

Thus the dimension of S^λ is 6. The hook length formula is an alternate way to calculate the dimension.

Definition 23. The hook length of a box in a Young tableau is the number of boxes occurring beneath that box or to the right in the Young diagram, counting the box itself.

The numbers in this Young diagram illustrate the hook lengths.



Proposition 24 (Hook Length Formula)**.** *The dimension of S^λ is $n!$ divided by the product of the hook lengths in the Young diagram of shape λ.*

For example, we can check again that the dimension of $S^{(3,1,1)}$ is $\frac{5!}{5\cdot 2\cdot 2} = 6$.

To combinatorially compute the character of S^λ, we either use the Frobenius formula (a generating function) or a recursive algorithm.

Theorem 25. *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of n and $\mu = (\mu_1, \ldots, \mu_m)$ be the cycle shape of $g \in S_n$. The character of S^λ evaluated on g is the coefficient of $x_1^{\lambda_1+l-1} x_2^{\lambda_2+l-2} \ldots x_l^{\lambda_l}$ in*

$$\prod_{1 \leq i \leq j \leq l} (x_i - x_j) \prod_{i=1}^{m} \left(x_1^{\mu_i} + \ldots + x_l^{\mu_i} \right).$$

The alternative method of calculating the character values is called the Murnaghan-Nakayama Rule [2, Section 4.10].

Definition 26. A rim hook is a connected chain of boxes of a Young diagram for which produce a valid Young diagram when they are removed.

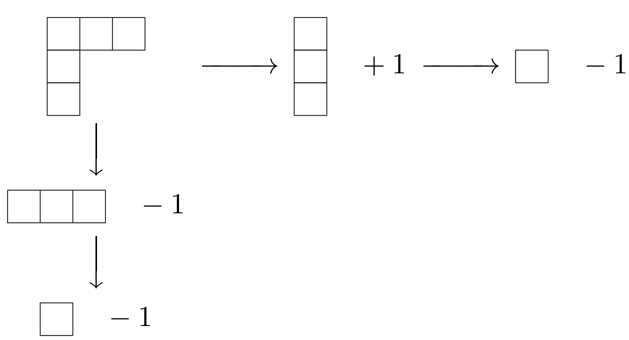
For example, in the first figure the boxes marked with dots are a rim hook, while in the second they are not:

Although removing the marked boxes seems to produce a Young diagram of the correct shape, it has an empty row on top: it is represented by the partition $(0, 2, 1)$. Define the height of a rim hook to be the number of rows the hook includes, minus 1. So the rim hook above has height1.

According to the Murnaghan-Nakayama rule, to evaluate the character of S^λ on g of cycle shape μ, consider all possible rim hooks consisting of μ_1 boxes of the Young diagram of shape λ. Remove them, producing several smaller Young diagrams. For each, associate a sign equal to negative one raised to the height of rim hook. Then recursively follow this procedure to calculate the character of a $S^{(\lambda_2,\ldots,\lambda_r)}$ on a cycle of shape $(\mu_2, \ldots, \mu_l)$. Add the results. Algebraically, this formula is expressed

$$\chi^\lambda(\mu) = \sum_{\xi : |\xi| = \mu_1} (-1)^{\text{height}(\xi)} \chi^{\lambda \backslash \xi}(\mu \backslash \mu_1)$$

For example, let $\lambda = (3, 1, 1)$ and $\mu = (2, 2, 1)$. Keeping track of the signs,



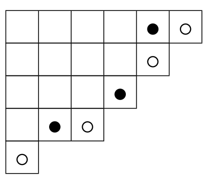
Since $\square$ is the trivial representation of S_1, the character is 1. Thus the character of $S^{(3,1,1)}$ on the cycle $(1\,2)(3\,4)$ is -2. It is similarly easy to calculate the value on the other conjugacy classes. Thus it becomes apparent that S^λ is the last character, of dimension 6, in the character table of S_5 that we previously found only via the orthogonality relations. Young tableaux have provided us with a systematic way to construct irreducible representations of S_n.

4. Understanding Induced Representations of S_n

Although induced representations in general seem hard to deal with, those for the symmetric group are very easy to analyze using Young tableau. The result is called the branching rule. To state it, we need to think about how to modify Young diagrams by adding or removing boxes.

Definition 27. If λ is a Young tableau, an inner corner of λ is a box that can be removed leaving a valid Young diagram. Denote the diagram formed by removing it by λ^-. An outer corner is a box that can be added to λ that produces a valid Young diagram. Denote the diagram formed by adding it λ^+.

For example, let $\lambda = (5, 4, 4, 2)$. Inner corners are marked by $\bullet$ and outer corners by $\circ$:



Note that the outer corners are not actually part of the diagram for λ.

Theorem 28. *If λ is a partition of n, then*

$$\mathrm{Res}^{S_n}_{S_{n-1}} S^\lambda = \bigoplus_{\lambda^-} S^{\lambda-} \quad \textit{and} \quad \mathrm{Ind}^{S_{n+1}}_{S_n} S^\lambda = \bigoplus_{\lambda^+} S^{\lambda^+}$$

where the sums run over all inner and outer corners respectively.

Proof. First, note that $\dim S^\lambda = \sum_{\lambda^-} \dim(S^{\lambda^-})$. This is simply because a basis is given by e_t for t a standard tableau, and every standard tableau has n in one of its inner corners. Removing it forms a standard tableau of shape λ^-. Let $r_1 < r_2 < \ldots < r_k$ be the rows with inner corners, and let λ^i be the diagram obtained by removing the inner corner in row r_i. Likewise, if t is a tableau of shape λ with n in the corner or row r_i, then t^i is the (standard) tableau obtained by removing n.

We will find a chain of S_{n-1} modules

$$0 = V_0 \subset V_1 \subset V_2 \ldots \subset V_k = S^\lambda$$

such that $V_i/V_{i-1} \simeq S^{\lambda^i}$ as S_{n-1} modules. As every complex representation of finite groups is semi-simple, all short exact sequences split so this implies the direct sum decomposition.

Take V_i to be the vector space spanned by e_t, where t is a standard tableau with n appearing at the end of one of the first i rows. Define $\theta_i : M^\lambda \to M^{\lambda^i}$ by sending a tabloid t to t^i if n is in row r_i of t, 0 otherwise. Extend by linearity. This is certainly a S_{n-1} homomorphism since n is fixed by S_{n-1}. For a standard tableau t, all of the elements appearing in the sum for e_t have an n in the same row as t or above it. Thus a little thought shows if n is in row r_i we have $\theta_i(e_t) = e_{t^i}$. Otherwise if n is in a higher row $\theta_i(e_t) = 0$. Remembering that $\{e_t\}$ for t a standard tableau of shape λ^i form a basis for S^{λ^i}, we conclude the image of V_i under θ_i is S^{λ^i}. Furthermore, V_{i-1} is included in the kernel of θ_i by definition. Thus we have a chain of S_{n-1} modules

$$0 \subset V_1 \cap \ker(\theta_1) \subset V_1 \subset V_2 \cap \ker(\theta_2) \subset V_2 \ldots \subset S^\lambda.$$

However, $V_i/(V_i \cap \ker(\theta_i)) \simeq \theta_i V_i = S^{\lambda_i}$. Since the dimensions of the S^{λ_i} add up precisely to the dimension of S^λ, we must have that $V_i \cap \ker(\theta_i) = V_{i-1}$ and hence that $V_i/V_{i-1} \simeq S^{\lambda_i}$. This concludes the proof of the first part of the theorem.

For the second, we use Frobenius reciprocity. Let χ^λ denote the character of S^λ. Write

$$\operatorname{Ind}_{S_n}^{S_{n+1}} S^\lambda = \bigoplus_{\lambda'} c_{\lambda'} S^{\lambda'}$$

where λ' runs over partitions of $n+1$. Then taking an inner product with S^μ, we see that

$$\begin{aligned} c_\mu &= (\operatorname{Ind}_{S_n}^{S_{n+1}} \chi^\lambda, \chi^\mu) \\ &= (\chi^\lambda, \operatorname{Res}_{S_n}^{S_{n+1}} \chi^\mu) \\ &= (\chi^\lambda, \sum_{\mu^-} \chi^{\mu^-}) \end{aligned}$$

using Frobenius reciprocity and the branching rule for restrictions. But this is 0 unless $\lambda = \mu^-$. Thus the result is 1 if $\mu = \lambda^+$ and 0 otherwise, so there is one copy of each S^{λ^+} in the induced representation and no other irreducibles. This completes the proof. $\square$

As an example, we induce two representations to S_4. First, we will show that the regular representation of S_4 is the representation induced from the trivial character of the trivial subgroup. This corresponds to the partition (1) of S_1. Inducing to S_2 we see that

$$\operatorname{Ind}_{S_1}^{S_2}(1) = S^{(1,1)} \oplus S^{(2)}.$$

Inducing again using the branching rule, we see

$$\operatorname{Ind}_{S_1}^{S_3}(1) = S^{(3)} \oplus (S^{(2,1)})^2 \oplus S^{(1,1,1)}.$$

Using the branching rule for a third time, we obtain

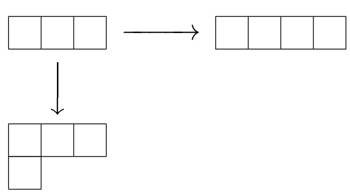
$$\operatorname{Ind}_{S_1}^{S_4}(1) = S^{(4)} \oplus (S^{(3,1)})^3 \oplus (S^{(2,1,1)})^3 \oplus (S^{(2,2)})^2 \oplus S^{(1,1,1,1)}.$$

This matches our theoretical knowledge of the regular representation: each irreducible appears with multiplicity equal to its dimension.

For a more interesting example, let us return to the example of calculating $\operatorname{Ind}_{S_3}^{S_4} 1$ which was non-trivial during the lecture on induced representations. The branching rule makes it apparent at a glance that

$$\operatorname{Ind}_{S_3}^{S_4} 1 = S^{(3,1)} \oplus S^{(4)}.$$

In other words, it is the permutation representation, the same answer as before. However, adding boxes to a Young diagram is much easier than carrying out the construction or computing tensor products.



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