Written Homework 7
MAC 2312
July 6, 2015

Question #1 Determine if the series converges or diverges.
[Assigned June 29, 2015, due in class June 30, 2015]

\[
\sum_{n=2}^{\infty} \frac{(n + 3)^{2n}}{(n^2 - 3)^{3n}}
\]

We proceed using the root test:

\[
\lim_{n \to \infty} \sqrt[n]{\left| \frac{(n + 3)^{2n}}{(n^2 - 3)^{3n}} \right|} = \lim_{n \to \infty} \frac{(n + 2)^2}{(n^2 - 3)^3} = 0 < 1
\]

Thus, the series converges absolutely by the root test.

Question #2 Determine if the series converges or diverges.
[Assigned June 30, 2015, due in class July 2, 2015]

\[
\sum_{n=0}^{\infty} \frac{2^{2n}}{n!}
\]

We proceed using the ratio test:

\[
\lim_{n \to \infty} \left| \frac{2^{2(n+1)}}{(n+1)!} \cdot \frac{n!}{2^{2n}} \right| = \lim_{n \to \infty} \frac{2^2}{n+1} = 0 < 1
\]

Hence, the series converges absolutely by the ratio test.

Question #3 Determine if the series converges or diverges.
[Assigned June 30, 2015, due in class July 2, 2015]

\[
\sum_{n=2}^{\infty} \frac{2^n}{e^{2n}}
\]

We proceed using the root test:

\[
\lim_{n \to \infty} \sqrt[n]{\left| \frac{2^n}{e^{2n}} \right|} = \lim_{n \to \infty} \frac{2}{e^2} = \frac{2}{e^2} < 1
\]
Hence, the series converges absolutely by the root test.

**Question #4** Determine the radius of convergence and the interval of convergence of the series. [Assigned July 2, 2015, due in class July 6, 2015]

\[
\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}
\]

We begin by using the ratio test:

\[
\lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-1)^n x^n} \right| = \lim_{n \to \infty} \frac{|x| n^2}{(n+1)^2} = |x| \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = |x|
\]

So by the ratio test, for this series to converge, we need \( |x| < 1 \). Hence, our radius of convergence is \( R = 1 \) and our interval of convergence, for now, is \((-1, 1)\). We need to check the endpoints:

\[ (x = -1) : \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \]

which is convergent by p-test.

\[ (x = 1) : \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \]

which is convergent by the alternating series test.

So our interval of convergence is \([-1, 1]\).

**Question #5** Determine the radius of convergence and the interval of convergence of the series. [Assigned July 2, 2015, due in class July 6, 2015]

\[
\sum_{n=1}^{\infty} n^n x^n
\]

We begin using the root test:

\[
\lim_{n \to \infty} \sqrt[n]{|n^n x^n|} = \lim_{n \to \infty} n|x| = \infty > 1
\]

So by the root test, the series diverges for all values of \( x \) except the center. Hence, the interval of convergence is \( \{0\} \) and the radius of convergence is 0.
Question #6 Determine the radius of convergence and the interval of convergence of the series. [Assigned July 2, 2015, due in class July 6, 2015]

\[ \sum_{n=0}^{\infty} \frac{(x - 2)^n}{n^2 + 1} \]

We begin by using the ratio test:

\[
\lim_{n \to \infty} \left| \frac{(x - 2)^{n+1}}{(n + 1)^2 + 1} \cdot \frac{n^2 + 1}{(x - 2)^n} \right| = \lim_{n \to \infty} \frac{|x - 2| (n^2 + 1)}{(n + 1)^2 + 1} = |x - 2| < 1
\]

So by the ratio test we have that this series converges for \( |x - 2| < 1 \). So our radius of convergence is 1 and our interval of convergence, for now, is \((1, 3)\). We must now check the endpoints:

\((x = 1) : \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + 1}\)

which converges by the alternating series test.

\((x = 3) : \sum_{n=0}^{\infty} \frac{1}{n^2 + 1}\)

which converges by comparison.

Hence, our interval of convergence is \([1, 3]\).

Question #7 Determine the radius of convergence and the interval of convergence of the series. [Assigned July 2, 2015, due in class July 6, 2015]

\[ \sum_{n=1}^{\infty} \frac{3^n (x - 4)^n}{\sqrt{n}} \]

We begin by using ratio test:

\[
\lim_{n \to \infty} \left| \frac{3^{n+1} (x - 4)^{n+1}}{\sqrt{n + 1}} \cdot \frac{\sqrt{n}}{3^n (x - 4)^n} \right| = \lim_{n \to \infty} \frac{3|x - 4| \sqrt{n}}{\sqrt{n + 1}} = 3|x - 4| < 1 \Rightarrow |x - 4| < \frac{1}{3}
\]

So by the ratio test we have that the series converges for \( |x - 4| < \frac{1}{3} \). So the radius of convergence is \( \frac{1}{3} \) and the interval of convergence, so far, is \( \left( \frac{11}{3}, \frac{13}{3} \right) \). We must now check the endpoints:
\[
\left( x = \frac{11}{3} \right) : \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}
\]

which converges by the alternating series test.

\[
\left( x = \frac{13}{3} \right) : \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}
\]

which diverges by the p-test.

Hence, the interval of convergence is \( \left[ \frac{11}{3}, \frac{13}{3} \right) \).