Lecture 14: Section 3.3
Rates of Change

**ex.** The boiling point \( B \) of water (in Fahrenheit degrees) \( h \) thousand feet above sea level is given by the function \( B(h) = -1.8h + 212 \).

What do the intercept and slope tell you?

By how much does \( B \) change starting at sea level if elevation rises by 1000 feet?

By how much does \( B \) change starting from a mile marker on a trail in the Rockies (5280 feet) if a hiker climbs another 1000 feet?

How do we measure rate of change if our function is not linear?
ex. Let \( p(x) = 20 - 0.02x \) be the demand function for a product.

1) Find the revenue function \( R(x) \).

\[ R(x) = \]

Consider its graph:
2) Use the graph to estimate the rate at which revenue is changing when

a) \( x = 100 \) by approximating the change on the interval \([100, 200]\)

b) \( x = 500 \)

c) \( x = 900 \) by approximating the change on the interval \([800, 900]\)

Can we find a more precise way of measuring rate of change at a specific \( x \)-value?
Consider again the revenue function 
\( R(x) = 20x - 0.02x^2 \) and the following table of values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>100</th>
<th>400</th>
<th>500</th>
<th>600</th>
<th>900</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R(x) )</td>
<td>1800</td>
<td>4800</td>
<td>5000</td>
<td>4800</td>
<td>1800</td>
</tr>
</tbody>
</table>

1) What is the change in revenue as production increases from 100 to 400 units?

2) What is the corresponding **average** change in revenue?
3) What is the **average** change in revenue as production increases from 600 to 900 units?

In general, for \( y = f(x) \),

The average rate of change of \( y \) with respect to \( x \) as \( x \) changes from \( a \) to \( b \) (on \([a, b]\)) is given by

\[ \frac{f(b) - f(a)}{b - a} \]
To find the rate at which $y$ is changing with respect to $x$ at a specific value $x = a$:

Consider the average rate of change of $y = f(x)$ with respect to $x$ as $x$ changes from $a$ to $a + h$:

What is the **instantaneous** rate of change of $y$ with respect to $x$ when $x = a$?
NOTE: If $a + h = b$ in the formula, then $h = b - a$ and as $h \to 0$, $b \to \underline{\phantom{0}}$.

We can rewrite our formula for the **instantaneous rate of change of** $f(x)$ **with respect to** $x$ **when** $x = a$:

$$
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}
$$

**ex.** 1) Find a formula for the rate of change of revenue $R(x) = 20x - 0.02x^2$ at the production level $x = 100$. 
2) For \( R(x) = 20x - 0.02x^2 \), what is the additional revenue if sales increase from \( x = 100 \) to \( x = 101 \)?

\[ (R(101) = 1815.98) \]

**Def. Marginal Revenue**

3) Can we find a general formula for the rate of change of revenue \( R(x) = 20x - 0.02x^2 \) at \( x = a \)?
4) For \( R(x) = 20x - 0.02x^2 \), find the marginal revenue when

a) \( x = 100 \)

b) \( x = 900 \)

What happens when \( x = 500 \)?

**Average, Instantaneous Velocity**

**ex.** A car travels 360 miles in 6 hours. What is its average velocity (speed)?
Let $s(t)$ be the function which gives the position of an object from some starting point at $t$ units of time. This is a **position function**.

**NOTE:** We use $h(t)$ to represent vertical motion.

Then **average velocity over** $[a, b]$ is

How do we find **instantaneous** velocity?

**NOTE:** velocity and speed
ex. Suppose that the distance (in feet) covered by a car moving along a straight road \( t \) seconds after starting from rest is given by the position function \( s(t) = 2t^2 + 30t \).

1) Find the average velocity of the car over the time interval \([40, 44]\).

2) Find the instantaneous velocity when \( t = 40 \).
Additional Example

**ex.** Suppose that the demand for a barrel of oil in a certain country is given by the formula

\[ D(p) = \frac{4000}{p}, \]

where \( D \) is the yearly demand per individual when the price per barrel is \( p \) dollars.

1) Find the average rate of change in demand when the price increases from \$80 to \$100.
2) Now find the rate at which the demand 

\[ D(p) = \frac{4000}{p} \] 

is changing when \( p = 80 \).

4) Find the rate at which demand is changing when the price is \( a \) dollars.
Now you try it! Problems are on pages 14 and 15.

1. The concentration in mg/ml of a certain drug in the bloodstream is given in the table below. \( C(t) \) gives the concentration of the drug at time \( t \) measured in minutes.

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
 t & 0 & 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\
 C(t) & 0 & 6 & 9 & 14 & 21 & 30 & 41 & 54 & 69 & 90 \\
\end{array}
\]

At what rate is the concentration changing over

(a) the first 5 minutes       (b) the last 10 minutes

(c) the 45 minute period

2. Suppose that for a certain piece of machinery the cost function is determined to be \( C(x) = 0.04x^2 + 2x + 40 \) where \( C \) is the cost in thousands of producing \( x \) items.

(a) Find the change in cost as production increases from 10 to 11 items (average rate of change of cost from \( x = 10 \) to \( x = 11 \)).

(b) Find the marginal cost when \( x = 10 \). Do this by finding a formula for the rate at which cost is changing when \( x = 10 \) (instantaneous rate of change of \( C(x) \) at \( x = 10 \)). Interpret your answer.

3. The displacement (distance in inches from the starting point) of an object moving in a straight line is given by the position function \( s(t) = t^2 - 3t - 4 \) where \( t \) is measured in seconds.

(a) Find the average velocity of the object in the first three seconds of travel \((t = 0 \) to \( t = 3 \)).

(b) Find the velocity at which the object is traveling at time \( t = 2 \).

(c) Find a formula for the instantaneous velocity of the object at time \( t = a \). When does the object first come to a stop?
4. The quarterly profit for a small business based on the amount spent on advertising is given by the formula \( P(x) = -\frac{1}{3}x^2 + 8x + 20 \) where \( P(x) \) is the quarterly profit in thousands of dollars when \( x \) thousand dollars is spent on advertising.

(a) Find the average rate of change in profit as the advertising budget increases from $6,000 per quarter (\( x = 6 \)) to $9,000 per quarter (\( x = 9 \)).

(b) Find a formula for the rate of change of profit at a given \( x = a \).

(c) Use your formula to find the rate at which quarterly profit is changing when the company spends $15,000 on advertising.

(d) Can you find the amount of money the business should spend on advertising to maximize its profit? Hint: that will occur when the rate of change of profit (see your formula from (c)) is equal to 0.

5. The population of a settlement \( t \) years after its founding is given by \( N(t) = \frac{300t}{1 + t}, t \geq 0 \).

(a) Find the population after 9 years.
(b) Find the average change in population from year 5 to year 9.
(c) Find the instantaneous growth rate at time \( t = 9 \).