Lecture 21: Section 6.5
Related Rates

We now consider applications of the Chain Rule and implicit differentiation that involve quantities changing with respect to time.

ex. A manufacturer has found that the daily profit from the sale of $x$ items is $P(x) = -5x^2 + 200x$. At the same time it has been determined that the number of items that can be produced is related to the number of hours the factory has been open that day, $t$, by the equation $x = t^2 + 2t$. Find the rate at which profit is changing with respect to time when the factory has been operating for 3 hours.
We now consider quantities, related to each other, that are both changing with respect to time. How are their rates of change related?

ex. A point is moving along the graph of the function \( y = \frac{1}{1 + x^2} \) so that the velocity in the \( x \) direction as it passes the point \((-2, \frac{1}{5})\) is 2 cm/min. What is the velocity in the \( y \) direction at this time? What about when \( x = 1 \), assuming its velocity does not change?
**ex.** Water from a water heater is leaking onto the basement floor of a home. A circular puddle is created whose area is increasing at the rate of $25\pi$ square inches per minute. How fast is the radius of the puddle increasing when it is 10 inches?
To solve a related rates problem:

1) Identify the desired rate of change; assign variables to all related quantities. Determine what rate(s) and information are given. Draw a sketch if possible.

2) Write an equation (mathematical model) relating the variables involved.

3) Differentiate the equation implicitly with respect to time.

4) Substitute known values and solve for the desired rate of change.
ex. A company is increasing its production of a new camera at the rate of 40 units per month. If the demand and cost functions are \( p(x) = 100 - \frac{3x}{100} \) and \( C'(x) = 1800 + 40x + .01x^2 \), at what rate is profit changing if current monthly sales are 800 units?
ex. A rope attached to the bow of a sailboat is drawn over a pulley 5 feet higher than the bow. If the boat is docking at the rate of 2.6 feet per second when there are 13 feet of rope out (13 feet between the boat and the pulley), how fast is the rope being drawn in? (Compare with problem #31, Sec. 6.5 in your text.)
ex. The demand function for a certain product is given by the equation $x^2p + 24p = 200$, where demand $x$ is measured in hundreds of units. At what rate is the demand for the product changing when the current weekly demand is 400 units and the price is decreasing at the rate of 25 cents per week?
ex. A liquid is to be cleared of sediment by pouring it through an inverted cone-shaped filter. The height of the cone is 16 inches and the diameter across the top is 8 inches, and as the liquid is poured out the height of the liquid in the filter remains four times the radius. If the liquid is flowing out at 2 cubic in/min, how fast is the depth of the liquid changing when it is 8 inches deep?
Additional Example

ex. A custom cabinetry shop estimates that the monthly output of the shop measured in thousands of dollars is given by $Q = 4x^{1/4}y^{3/4}$ where $x$ is the amount spent on labor and $y$ is the amount spent on capital (materials, fixed costs, etc.), both measured in thousands. (See Lecture 20, page 8).

Now suppose that the output remains constant, but capital expenses are rising by $250$ per month when $16,000$ is spent on labor and $81,000$ on capital per month. At what rate will labor expenses be changing with respect to time?
Now You Try It! Problems are on pages 10 and 11.

1. Suppose that a point is moving along the graph of \( xy^2 + 2y = -4 \) so that its \( x \)-coordinate is increasing at the rate of 1.5 inches per minute. How fast is the \( y \)-coordinate changing with respect to time when \( y = -1 \)?

2. A department store estimates that weekly sales \( S \) and weekly advertising costs \( x \) are related by the equation \( S = 4000 + 60x + 0.02x^2 \). Right now weekly advertising costs are $2000, but these costs are rising at a rate of $8 per week. Find the rate at which the current weekly sales are changing with respect to time.

3. A small spherical balloon is inserted into a blocked artery and is inflated at the rate of 0.0001\( \pi \) cubic millimeters per minute. How fast is the radius of the balloon increasing when that radius is 0.004 mm? Note that the volume of a sphere is given in terms of its radius according to the formula \( V(r) = \frac{4}{3}\pi r^3 \).

4. The demand and cost functions for a product are \( p(x) = 120 - 0.01x \) and \( C(x) = 40x + 1200 \), where \( x \) is the number of units produced weekly. If the manufacturer decides to increase production by 70 units per week, find the rate at which profit is changing with respect to time when the weekly production is 3600 units.

5. An observer is standing 50 feet from a helicopter launch pad. A helicopter lifts off vertically and rises at a rate of 40 ft/sec. At what rate is the distance between the observer and the helicopter changing at the instant when the helicopter is 120 feet high?

6. Sand is falling in a conical pile so that its height remains 1.5 times its radius. How fast is the volume of the sand in the pile changing at the instant when the pile is 6 feet high if the height is increasing by 0.5 feet per second at that time?
7. The base of a triangle is increasing at a rate of 2 ft/sec while the height is decreasing at a rate of 4 ft/sec. How fast is the area of the triangle changing when the base is 6 feet and the height is 10 feet? Hint: use the Product Rule.

8. The quantity demanded per month, $x$, of a new product is related to the unit price $p$ by the demand equation

$$x = f(p) = 12\sqrt{800,000 - p^2}.$$  

It is also estimated that the price of the product is given by

$$p(t) = \frac{2000}{8 + t} + 600, \quad 0 \leq t \leq 9,$$

where $t$ is in months. Find the rate at which the quantity demanded per month will be changing 2 months from now.