MAS 4301—Practice Problems for Exam #1

1. (a) Give an example of a finite abelian group which isn’t cyclic.
   (b) Set \( \mathbb{Z}_{\geq 0} = \{ n \in \mathbb{Z} : n \geq 0 \} \). Explain why \( \mathbb{Z}_{\geq 0} \) with the operation of addition is not a group.
   (c) Give an example of an infinite group \( G \) and an element \( x \in G \) such that \( x \) has finite order but \( x \neq e \).
   (d) Let \( S \) be a set. Give the definition of a binary operation on \( S \).

2. Let \( G \) be a group.
   (a) Suppose \( x, y, z \in G \) satisfy \( xz = yz \). Prove from first principles that \( x = y \).
   (b) Define what it means for \( G \) to be abelian.
   (c) Suppose that \( x^2 = e \) for every \( x \in G \). Prove that \( G \) is abelian. (Hint: First prove that \( x^{-1} = x \) for all \( x \in G \).)

3. Let \( G \) be a group and let \( a \in G \).
   (a) Define the centralizer \( C_G(a) \) of \( a \).
   (b) Prove that \( C_G(a) \subset C_G(a^2) \).
   (c) Prove that if \( G \) is abelian then the set \( H = \{ x^8 : x \in G \} \) is a subgroup of \( G \).

4. Let \( G = \langle x \rangle \) be a cyclic group of order 18.
   (a) For each \( n \) such that \( 1 \leq n \leq 7 \), list all elements of \( G \) with order \( n \).
   (b) Find \( m \) such that \( \langle x^{12} \rangle = \langle x^m \rangle \) and \( m \mid 18 \).
   (c) List the subgroups of \( G \). Your list should not have any repeats.

5. (a) Give an example of a group \( G \) and a subset \( S \subset G \) such that
      i. \( e \in S \),
      ii. \( S \) is closed under multiplication,
      iii. \( S \) is not a subgroup of \( G \).
      (Hint: \( G \) and \( S \) must be infinite.)
   (b) Give an example of a group \( G \) and \( x, y \in G \) such that \( (xy)^2 \neq x^2y^2 \).
   (c) Let \( G \) be a group and let \( H \subset G \). State the one-step subgroup test which gives sufficient conditions for \( H \) to be a subgroup of \( G \).

6. (a) Give an example of an abelian group \( G \) which is not cyclic.
   (b) Let \( G \) be a group and let \( x, y \in G \). Prove that \( (xy)^{-1} = y^{-1}x^{-1} \).
   (c) Let \( G \) be a group. Prove that \( G \) is abelian if and only if \( (xy)^{-1} = x^{-1}y^{-1} \) for all \( x, y \in G \).
7. Let $G$ be a group and let $x \in G$.
   
   (a) Define the order $|x|$ of $x$.
   (b) Prove from first principles that if $a \in \mathbb{Z}$ then $|x^a| \leq |x|$.
   (c) Suppose $x$ has finite order $n$. Prove that for every $y \in \langle x \rangle$ there is a unique integer $r$ such that $0 \leq r \leq n - 1$ and $y = x^r$.

8. Let $G = \langle x \rangle$ be cyclic group of order 30.
   
   (a) List the subgroups of $G$. Your list should not have any repeats.
   (b) Compute the order of $x^{14} \in G$.
   (c) Find all the elements of $G$ which have order 10.

9. (a) Explain why the set $\mathbb{R}_{\geq 0}$ of all nonnegative real numbers with the operation of addition is not a group.
   (b) Let $G$ be a group and let $x \in G$. Prove the uniqueness of the inverse of $x$: If $y, y'$ are elements of $G$ such that $xy = yx = e$ and $xy' = y'x = e$ then $y = y'$.
   (c) Let $G$ be a group with the property that for every $x, y, z \in G$, if $xy = zx$ then $y = z$. Prove that $G$ is abelian.

10. Let $G$ be a group.
    
    (a) Give the definition of a subgroup of $G$.
    (b) Give an example of a group $G$ and two subgroups $H, K$ of $G$ such that $H \cup K$ is not a subgroup of $G$.
    (c) Let $H$ and $K$ be subgroups of $G$. Prove that $H \cap K$ is a subgroup of $G$.

11. Let $G$ be a group and let $a \in G$.
    
    (a) Give the definition of $\langle a \rangle$.
    (b) Prove that $\langle a \rangle$ is a subgroup of $G$.
    (c) Prove that $C_G(a) = \{x \in G : ax = xa\}$ is a subgroup of $G$.

12. (a) Give the definition of a cyclic group.
    (b) Find every $a \in \mathbb{Z}_{24}$ such that $\langle a \rangle = \mathbb{Z}_{24}$.
    (c) Let $G = \langle x \rangle$ be a cyclic group of order 28. Find all the subgroups of $G$. Your list should not have any repeats.