MAS 4105—Practice Problems for Exam #1

1. In each case determine whether $W$ is a subspace of $\mathbb{R}^3$. If $W$ is a subspace check that the subspace conditions are satisfied. If $W$ is not a subspace give a specific example which shows that $W$ violates one of these conditions.

(a) $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 - 4x_2 - x_3 = 0 \right\}$

(b) $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1^2 - x_2^2 = 0 \right\}$

2. Let $V$ be a vector space over $F$, and assume there is a finite set $S$ such that $\text{Span}(S) = V$.

(a) Give the definition of the dimension of $V$.

(b) Let $d$ be the dimension of $V$ and let $T$ be a subset of $V$ with $d$ elements. Must $T$ be a basis for $V$? Give a short explanation for your answer.

(c) Let $\vec{y} \in V$ be such that $\vec{y} \notin S$. Is $S \cup \{\vec{y}\}$ linearly independent? Give a short justification for your answer. (One possible answer is “Can’t tell from the given information”.)

3. Find a basis $\beta$ for each vector space $V$. No justification is required. (You may assume that $V$ is a vector space in each case.)

(a) $M_{1 \times 4}(\mathbb{R})$

(b) $V = \{ f \in P_2(\mathbb{R}) : f(3) = 0 \}$

(c) $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + 3x_3 = 0 \right\}$

4. (a) Determine, with justification, whether $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

(b) Let $V$ be a vector space over a field $F$ and let $\{\vec{x}, \vec{y}\}$ be a basis for $V$. Let $c, d$ be nonzero elements of $F$. Prove that $\{c\vec{x}, d\vec{y}\}$ is a basis for $V$.

5. Find a basis $\beta$ for each vector space $V$. No justification is required.

(a) $V = P_3(\mathbb{R})$

(b) $V = \{ A \in M_{2 \times 2}(\mathbb{R}) : A^t = -A \}$

(c) $V = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + 2x_2 + 3x_3 = 0 \}$
6. In each case determine whether $W$ is a subspace of $P_3(\mathbb{R})$. If $W$ is a subspace check that the subspace conditions are satisfied. If $W$ is not a subspace give a specific example which shows that $W$ violates one of these conditions.

(a) $W = \{ f \in P_3(\mathbb{R}) : f(4) \geq 0 \}$

(b) $W = \{ f \in P_3(\mathbb{R}) : f(0) = 0 \text{ and } f(3) = 0 \}$

7. (a) Determine, with justification, whether the set $S = \{ 1 - X, 1 - X^2, X - X^2 \}$ is a basis for $P_2(\mathbb{R})$.

(b) Let $V$ be a vector space over a field $F$ and let $\{ \vec{x}, \vec{y} \}$ be a basis for $V$. Prove that $\{ \vec{x}, \vec{x} + \vec{y} \}$ is a basis for $V$.

8. No justification is needed for this problem.

(a) Let be a vector space over the field $F$ and let $S = \{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \}$ be a subset of $V$. Define what it means for $S$ to be linearly independent.

(b) Let $S = \{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \}$ be a set of vectors in $V$. Define $\text{Span}(S)$.

(c) Let $V$ be a vector space, let $S = \{ \vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_5 \}$ be a linearly independent subset of $V$, and let $S_0 = \{ \vec{x}_1, \vec{x}_2, \vec{x}_3 \}$. Answer the following questions about $S_0$. (Possible answers: Yes, No, Can't Tell from the given information.)
   i. Is $S_0$ linearly independent?
   ii. Does $S_0$ span $V$?
   iii. Is $S_0$ a basis for $V$?

9. Find a basis $\beta$ for each vector space $V$. (No justification needed for this problem.)

(a) $V = M_{2\times2}(F)$

(b) $V = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 - 5x_3 = 0 \}$

(c) $V = \{ f \in P_2(F) : f(0) = 0 \}$

10. In each case determine whether $W$ is a subspace of $\mathbb{R}^3$. If $W$ is a subspace check that the subspace conditions are satisfied. If $W$ is not a subspace show that $W$ violates one of these conditions.

(a) $W = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - 5x_2 = 0 \}$

(b) $W = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 \geq 0 \}$

11. (a) Determine whether the set $S = \{ (1, 1, 2), (0, 1, 1), (1, 1, 3) \}$ is a basis for $\mathbb{R}^3$. Explain your reasoning, and be sure to use Gaussian elimination if you need to solve a system of linear equations.

(b) Let $S = \{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \}$ be a subset of $V$. Prove that if $\vec{y} \in \text{Span}(S)$ then the set $S \cup \{ \vec{y} \} = \{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n, \vec{y} \}$ is linearly dependent.

12. (a) Let $S = \{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \}$ be a set of vectors in $V$. Define what it means for $S$ to be a basis for $V$. 

(b) Find three vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ in $\mathbb{R}^3$ such that the set $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is linearly dependent, but none of the three vectors is a scalar multiple of any of the others.

(c) Let $S = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ be a linearly dependent set of vectors in $V$, and let $T \subset S$. Does $T$ have to be linearly dependent as well? If your answer is yes give a short explanation. If your answer is no give an example.

13. Define $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T(f) = (x + 1)f'(x) - f(x)$.

(a) Prove that $T$ is a linear transformation.

(b) Find a basis for $R(T)$.

(c) Find a basis for $N(T)$.

14. Define $T : F^3 \to P_2(F)$ by setting

$$T(a_1, a_2, a_3) = (a_1 + 2a_3) + 0x + (a_2 - a_3)x^2.$$ 

(a) Prove that $T$ is a linear transformation.

(b) Find a basis for $R(T)$.

(c) Find a basis for $N(T)$.

15. Let $V$ and $W$ be vector spaces with $\dim(V) = n$ and $\dim(W) = m$. Let $T : V \to W$ be a linear transformation.

(a) Define the rank of $T$ and the nullity of $T$.

(b) Give a formula which relates $\text{rank}(T)$ to $\text{nullity}(T)$.

(c) Suppose that $\dim(V) = 11$, $\dim(W) = 19$, and $\dim(\text{N}(T)) = 8$. Find the dimension of $R(T)$. 