MAS 5311: Practice problems for Exam #1

1. Let \( \phi : G \to H \) be a homomorphism and let \( E \) be a subgroup of \( H \). Prove that the set \( \phi^{-1}(E) = \{ g \in G : \phi(g) \in E \} \) is a subgroup of \( G \).

2. Let \( G \) be a group and let \( A \) be a set.
   (a) Give the definition of an action of \( G \) on \( A \).
   (b) Let \( G \) act on \( A \) via \( (g, a) \mapsto g \cdot a \). For \( g \in G \) define \( \sigma_g : A \to A \) by \( \sigma_g(a) = g \cdot a \).
   Prove that \( \sigma_g \) is a permutation of \( A \).

3. Let \( G = D_{16} = \langle s, r : s^2 = r^8 = 1, rs = sr^{-1} \rangle \) be the dihedral group of order 16.
   (a) Compute the centralizers \( C_G(\{s\}) \) and \( C_G(\{r, r^{-1}\}) \).
   (b) Compute the normalizers \( N_G(\{s\}) \) and \( N_G(\{r, r^{-1}\}) \).

4. Let \( G = \langle x \rangle \) be a cyclic group of order 20. Draw the complete lattice of subgroups for \( G \). (No justification needed for this problem.)

5. (a) Prove that if \( G \) is an abelian group and \( H \) is a subgroup of \( G \) then \( H \trianglelefteq G \).
   (b) Let \( G \) be a group and let \( N \) be a normal subgroup of \( G \) such that \( G/N \) is abelian. Let \( H \) be a subgroup of \( G \) such that \( H \supseteq N \). Prove that \( H \trianglelefteq G \).
   (You may want to use a part of the Fourth Isomorphism Theorem.)

6. (a) State the First Isomorphism Theorem.
   (b) Prove the Second Isomorphism Theorem: If \( H \trianglelefteq G \), \( K \subseteq G \), and \( H \trianglelefteq N_G(K) \) then \( HK/K \cong H/(H \cap K) \). You may assume \( HK \subseteq G \) and \( K \trianglelefteq HK \).

7. Let \( G = GL_2(\mathbb{R}) \) act on \( \mathbb{R}^2 \) by matrix multiplication: \( A \cdot \vec{v} = A\vec{v} \) for \( A \in G \), \( \vec{v} \in \mathbb{R}^2 \).
   (a) Determine the stabilizer \( G_{\vec{e}_1} \) of \( \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^2 \).
   (b) Determine the stabilizer \( G_{\vec{e}_2} \) of \( \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2 \).
   (c) Determine the kernel of the action of \( G \) on \( \mathbb{R}^2 \).

8. Let \( G \) and \( H \) be cyclic groups with the same finite order \( n \). Prove that \( G \cong H \).

9. Give the complete lattice of subgroups of the cyclic group \( Z_{30} \). (No proof needed.)

10. Let \( G \) be a group, let \( \{N_\lambda : \lambda \in \Lambda\} \) be a collection of normal subgroups of \( G \), and set \( S = \bigcup_{\lambda \in \Lambda} N_\lambda \). Prove that \( \langle S \rangle \) is a normal subgroup of \( G \). (You may assume that \( \langle S \rangle \) is a subgroup of \( G \).)
11. We say that the group $G$ is finitely generated if there is a finite set $S$ such that $G = \langle S \rangle$.
   (a) Prove that if $H$ is a finitely generated subgroup of $(\mathbb{Q}, +)$ then $H$ is cyclic.
   (b) Prove that $(\mathbb{Q}, +)$ is not finitely generated.

12. Let $\phi : G \to H$ be a group homomorphism. Prove $G / \ker \phi \cong \phi(G)$. (You may assume $\ker \phi \triangleleft G$ and $\phi(G) \leq H$.)

13. Let $\phi : G \to H$ be a group homomorphism. Prove that $\phi$ is one-to-one if and only if $\ker(\phi)$ is trivial.

14. Let $G$ be an abelian group. Prove that the set $T = \{ g \in G : |g| < \infty \}$ is a subgroup of $G$.

15. Let $n \geq 3$ and recall that $D_{2n} = \langle r, s \rangle$ is a group of order $2n$, with $r^n = s^2 = 1$ and $srs^{-1} = r^{-1}$. Prove that $Z(D_{2n})$ is trivial if and only if $n$ is odd.

16. Prove Lagrange’s theorem: Let $G$ be a finite group and let $H \leq G$. Then $|G| = |G : H| \cdot |H|$.

17. Prove that if $G$ is an abelian simple group then $G \cong Z_p$ for some prime $p$. (Do not assume that $G$ is finite.)

18. (a) Define what it means for a group $G$ to be solvable.
   (b) Let $G$ be a solvable group and let $H$ be a subgroup of $G$. Prove that $H$ is solvable.

19. Let $G$ be a group and let $N$ be a normal subgroup of $G$ such that both $G/N$ and $N$ are solvable. Prove that $G$ is solvable.