MAS 4301—Practice Problem Set #2

1. Let \( A = \mathbb{Z}_{10} = \{0, 1, 2, \ldots, 9\} \) and define \( \sigma : A \to A \) by \( \sigma(a) = a + 4 \) (mod 10). (So \( \sigma(a) \) is \( a + 4 \) modulo 10.)
   
   (a) Prove that \( \sigma \) is a permutation of \( A \).
   (b) Find a cycle decomposition for \( \sigma \).
   (c) Express \( \sigma \) as a product of 2-cycles.
   (d) Determine whether the permutation \( \sigma \) is even or odd.

2. (a) Determine the number of elements of order 4 in \( S_5 \).
   (b) Determine the number of elements of order 4 in \( A_5 \).
   (c) Let \( G = \langle x \rangle \) be a cyclic group of order 20. Determine all automorphisms of \( G \). In particular, for each \( \sigma \in \text{Aut}(G) \) give the value of \( \sigma(x) \).
   (d) Let \( A \) be a set, let \( x \in A \), and let \( G \) be a subgroup of the symmetric group \( S_A \). Define the stabilizer \( \text{Stab}_G(x) \) of \( x \) in \( G \).

3. (a) Let \( S_4 \) be the symmetric group and let \( D_{12} \) be the dihedral group. Prove or disprove that \( S_4 \) is isomorphic to \( D_{12} \). (Note that both groups have order 24.)
   (b) Let \( \phi : G \to \overline{G} \) be an isomorphism. Prove that for all \( x \in G \) we have \( \phi(x^{-1}) = \phi(x)^{-1} \). You may use the formula \( \phi(e) = \overline{e} \), where \( e \) is the identity of \( G \) and \( \overline{e} \) is the identity of \( \overline{G} \).
   (c) Let \( \phi \) be an automorphism of \( G \). Prove that the set \( H = \{ x \in G : \phi(x) = x \} \) is a subgroup of \( G \).

4. Let \( G \) be a finite group.
   
   (a) Let \( H \leq G \). State Lagrange's theorem for \( G \) and \( H \).
   (b) Prove that if \( x \in G \) then \( |x| \) divides \( |G| \).
   (c) Let \( H, K \) be subgroups of \( G \), with \( H \subset K \). Use Lagrange's theorem to prove that \( |G : H| = |G : K| \cdot |K : H| \).

5. Let \( A = U(11) = \{1, 2, \ldots, 10\} \) and define \( \sigma : A \to A \) by \( \sigma(a) = 3 \cdot_a 11 \). (In other words, \( \sigma(a) \) is equal to \( 3a \) modulo 11.)
   
   (a) Prove that \( \sigma \) is a permutation of \( A \).
   (b) Find a cycle decomposition for \( \sigma \).
   (c) Express \( \sigma \) as a product of 2-cycles.
   (d) Determine whether the permutation \( \sigma \) is even or odd.
6. (a) Let \( H \) be a subgroup of the symmetric group \( S_n \). Prove that either all the elements of \( H \) are even permutations or exactly half of the elements of \( H \) are even permutations.

(b) Give an example of each of these, or explain why no example exists.

i. An element \( \sigma \) of \( S_5 \) which has order 6.

ii. An element \( \tau \) of \( S_6 \) which has order 7.

iii. An element \( \rho \) of \( A_9 \) which has order 10.

7. (a) Let \( A_4 \) be the alternating group and let \( D_6 \) be the dihedral group. Prove or disprove that \( A_4 \) is isomorphic to \( D_6 \).

(b) Let \( \phi : G \to \overline{G} \) be an isomorphism, let \( e \) be the identity of \( G \), and let \( \overline{e} \) be the identity of \( \overline{G} \). Prove from first principles that \( \phi(e) = \overline{e} \).

(c) Let \( \phi : G \to \overline{G} \) be an isomorphism and let \( x \in G \). Prove that if \(|x| = n\) then \(|\phi(x)| = n\). (You may assume that \( \phi(x^k) = \phi(x)^k \) for all \( k \in \mathbb{Z} \).)

8. Let \( G \) be a group and let \( a, b \in G \).

(a) Give the definition of the inner automorphism \( \phi_a : G \to G \) associated to \( a \).

(b) Prove that \( \phi_{ab} = \phi_a \circ \phi_b \).

(c) Prove that the set \( \text{Inn}(G) = \{ \phi_a : a \in G \} \) is a subgroup of \( \text{Aut}(G) \). (You may assume that the set \( \text{Aut}(G) \) of automorphisms of \( G \) is a group with the operation of composition.)

9. (a) Determine which of the groups \( \mathbb{Z}_2 \oplus \mathbb{Z}_3 \), \( S_3 \), \( A_4 \), \( \mathbb{Z}_6 \) are isomorphic to which, with some explanation.

(b) Let \( \phi : G \to \overline{G} \) be an isomorphism and let \( x \in G \). Prove that \( \phi(x^{-1}) = \phi(x)^{-1} \). You may assume that \( \phi(e_G) = \overline{e_G} \).

(c) Let \( \phi : G \to \overline{G} \) be an isomorphism and let \( H \) be a subgroup of \( G \). Prove that \( \phi(H) = \{ \phi(x) : x \in H \} \) is a subgroup of \( \overline{G} \).

10. (a) Let \( \mathbb{R} \) denote the group of real numbers with the operation of addition. Prove that the function \( \phi : \mathbb{R} \to \mathbb{R} \) defined by \( \phi(x) = 3x \) is an automorphism of \( \mathbb{R} \).

(b) Let \( G \) be a group of order \( n \), with \( 1 < n < \infty \). Prove that if the only subgroups of \( G \) are \( \{e_G\} \) and \( G \) then \( n \) is prime.

11. (a) Let \( G \) be a group and let \( H \) be a subgroup of \( G \). Give the definition of a left coset of \( H \) in \( G \).

(b) List the distinct left cosets of \( H = \langle (1 \, 2) \rangle \) in \( S_3 \).

(c) Give an example of a left coset \( L \) of \( \langle (1 \, 2) \rangle \) in \( S_3 \) and a right coset \( R \) of \( \langle (1 \, 2) \rangle \) in \( S_3 \) such that \( L \cap R \neq \{ \} \) and \( L \neq R \).

12. Let \( G \) be a finite group, let \( H \leq G \), and let \( x, y \in G \).
(a) Prove that if $xH \cap yH$ is nonempty then $xH = yH$.
(b) Prove that $xH$ has the same cardinality as $H$.
(c) Prove Lagrange’s Theorem: $|G| = |G : H| \cdot |H|$.