MHF 3202: Solutions to Practice Problem Set #3

1. (a) \( R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\} \).

(b) \( S \circ R = \{(x, z) \in A \times C \mid \exists y \in B((x, y) \in R \land (y, z) \in S)\} \)

(c) We use double containment. Let \((z, x) \in (S \circ R)^{-1}\). Then \((x, z) \in S \circ R\). Hence there is \(y \in B\) such that \((x, y) \in R\) and \((y, z) \in S\). It follows that \((y, x) \in R^{-1}\) and \((z, y) \in S^{-1}\). Hence \((z, x) \in R^{-1} \circ S^{-1}\). Thus \((S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1}\).

On the other hand, let \((z, x) \in R^{-1} \circ S^{-1}\). Then there is \(y \in B\) such that \((z, y) \in S^{-1}\) and \((y, x) \in R^{-1}\). Then \((y, z) \in S\) and \((y, x) \in R\). Hence \((x, z) \in S \circ R\). It follows that \((z, x) \in (S \circ R)^{-1}\). Thus \(R^{-1} \circ S^{-1} \subseteq (S \circ R)^{-1}\). We conclude that \(R^{-1} \circ S^{-1} = (S \circ R)^{-1}\).

2. (a) \( R \) is symmetric means \( \forall x \in A \forall y \in A(x Ry \rightarrow y Rx) \).

(b) \( \rightarrow \): Suppose \( R \) is symmetric. Let \((x, y) \in R\). Since \( R \) is symmetric we get \((y, x) \in R\). Hence \((x, y) \in R^{-1}\). Therefore \( R \subseteq R^{-1}\). Let \((y, x) \in R^{-1}\). Then \((x, y) \in R\). Since \( R \) is symmetric we get \((y, x) \in R\). Therefore \( R^{-1} \subseteq R\). Using double containment we conclude that \( R^{-1} = R\).

\( \leftarrow \): Suppose \( R^{-1} = R\). Let \((x, y) \in R\). Then \((y, x) \in R^{-1}\). Since \( R^{-1} = R\) we get \((y, x) \in R\). Hence \( R \) is symmetric.

(c) Let \( R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}\). Then \( R \) is reflexive and symmetric. Since \((1, 2) \in R\) and \((2, 3) \in R\) but \((1, 3) \notin R\), \( R \) is not transitive.

3. (a) \( R \) is antisymmetric means \( \forall x \in A \forall y \in A(x Ry \wedge y Rx \rightarrow x = y) \).

(b) Let \((x, y) \in R\). Since \( R \) is symmetric we get \((y, x) \in R\). Since \( R \) is antisymmetric it follows that \(x = y\). Hence \((x, y) = (x, x)\). Since \((x, x) \in i_A\) we get \((y, x) \in i_A\). Therefore \( R \subseteq i_A\).

(c) We are looking for the subsets of \( \{1, 2, 3, 4\}\) which are contained in both \( \{1, 2, 3\}\) and \( \{2, 3, 4\}\). These are just the subsets of \( \{1, 2, 3\} \cap \{2, 3, 4\} = \{2, 3\}\). So the lower bounds of \( B \) are \( \{2, 3\}, \{2\}, \{3\}, \) and \( \emptyset \).

4. (a) \([a]_R = \{x \in A \mid aRx\}\)

(b) Let \( x \in [b]_R\). Then \((x, b) \in R\). Since \( b \in [a]_R\) we have \((b, a) \in R\). Since \( R \) is transitive we get \((x, a) \in R\). Hence \( x \in [a]_R\). It follows that \([b]_R \subseteq [a]_R\). Let \( x \in [a]_R\).

Then \((x, a) \in R\). Since \( b \in [a]_R\) we have \((b, a) \in R\). Since \( R \) is symmetric we get \((a, b) \in R\). Since \( R \) is transitive it follows that \((x, b) \in R\). Hence \( x \in [b]_R\). It follows that \([a]_R \subseteq [b]_R\). It follows from double containment that \([b]_R = [a]_R\).

(c) \( R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}\)

5. (a) \( S \circ R = \{(x, z) \in A \times C \mid \exists y \in B((x, y) \in R \land (y, z) \in S)\} \)

(b) We prove the equivalent statement that \( S \circ R \neq \emptyset \) if and only if \( \text{Ran}(R) \cap \text{Dom}(S) \neq \emptyset \). Suppose \( S \circ R \neq \emptyset \). Then there is \((x, z) \in S \circ R\). Hence there is \( y \in B\) such that \((x, y) \in R\) and \((y, z) \in S\). Therefore \( y \in \text{Ran}(R)\) and \( y \in \text{Dom}(S)\), so \( y \in \text{Ran}(R) \cap \text{Dom}(S)\). Hence \( \text{Ran}(R) \cap \text{Dom}(S) \neq \emptyset \).
Conversely suppose Ran(R) \cap Dom(S) \neq \emptyset. Then there is y \in Ran(R) \cap Dom(S), so y \in Ran(R) and y \in Dom(S). Therefore there are x \in A and z \in C such that (x, y) \in R and (y, z) \in S. Hence (x, z) \in S \circ R, so S \circ R \neq \emptyset.

6 (a) R is transitive means \forall x \in A \forall y \in A \forall z \in A((xRy \land yRz) \rightarrow xRz).

(b) Suppose R is transitive. Let (x, z) \in R \circ R. Then there is y \in A such that (x, y) \in R and (y, z) \in R. By the transitive property of R we get (x, z) \in R. Hence R \circ R \subseteq R. Suppose conversely that R \circ R \subseteq R. Let x, y, z \in A be such that (x, y) \in R and (y, z) \in R. Then (x, z) \in R \circ R, so (x, z) \in R. It follows that R is transitive in this case.

(c) R_1 = \{(1, 2)\} and R_2 = \{(2, 3)\} are transitive on A, but R_1 \cup R_2 = \{(1, 2), (2, 3)\} is not, since (1, 3) \notin R_1 \cup R_2.

7 (a) i. The singletons \{1\}, \{2\}, \{3\}, and \{4\} are the minimal elements of A.

ii. Since A has more than 1 minimal element, it has no smallest element.

(b) i. Since b is the smallest element of B, for all x \in B we have bRx. This means that b is a lower bound for B.

ii. Let l \in A be a lower bound for B. Then lRx holds for every x \in B. In particular, since b \in B we have lRb. Therefore b is the greatest lower bound of B.

8 (a) \mathcal{F} is a partition of A means that \cup \mathcal{F} = A, \emptyset \notin \mathcal{F}, and \mathcal{F} is pairwise disjoint.

(b) [a]_R = \{x \in A \mid aRx\}

(c) Let [a]_R \in A/R. By definition we have [a]_R \subseteq A. Therefore \cup(A/R) \subseteq A. On the other hand, since R is reflexive we have a \in [a]_R for every a \in A. Therefore A \subseteq \cup(A/R). Hence \cup(A/R) = A. Since a \in [a]_R for every a \in A we also get \emptyset \notin A/R.

To show that A/R is pairwise disjoint, suppose there are a, b \in A such that [a]_R \cap [b]_R \neq \emptyset. Then there is c \in [a]_R \cap [b]_R. Hence c \in [a]_R and c \in [b]_R, so we have aRC and bRc. Therefore cRb by symmetry and aRb by transitivity. We claim that [a]_R = [b]_R. Let x \in [b]_R. Then bRx, and hence aRb by transitivity. Thus x \in [a]_R, so we have [b]_R \subseteq [a]_R. On the other hand, suppose x \in [a]_R. Then aRx, and hence xRa by symmetry. Hence by transitivity we get xRb. By symmetry we get bRx, so x \in [b]_R. Hence [a]_R \subseteq [b]_R. Therefore [a]_R = [b]_R. It follows that A/R is pairwise disjoint. We conclude that A/R is a partition of A.

9 We use double containment. Let (x, y) \in A \times (B \cup C). Then x \in A and y \in B \cup C. Therefore either y \in B or y \in C. In the first case we get (x, y) \in A \times B, and in the second case we get (x, y) \in A \times C. Hence (x, y) \in (A \times B) \cup (A \times C) in either case. Thus A \times (B \cup C) \subseteq (A \times B) \cup (A \times C).

On the other hand, suppose (x, y) \in (A \times B) \cup (A \times C). Then either (x, y) \in A \times B or (x, y) \in A \times C. In the first case we have x \in A and y \in B, and in the second case we have x \in A and y \in C. Hence x \in A and y \in B \cup C in either case, so we get (x, y) \in A \times (B \cup C). Thus (A \times B) \cup (A \times C) \subseteq A \times (B \cup C). We conclude that
\[ A \times (B \cup C) = (A \times B) \cup (A \times C). \]

10  (a)  We use the method of double containment. Suppose \((x, y) \in A \times (B \setminus C)\). Then \(x \in A\) and \(y \in B \setminus C\), so \(y \in B\) and \(y \notin C\). Since \(x \in A\) and \(y \in B\) we get \((x, y) \in A \times B\). Since \(y \notin C\) we get \((x, y) \notin A \times C\). Hence \((x, y) \in (A \times B) \setminus (A \times C)\). Therefore \(A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)\).

Suppose \((x, y) \in (A \times B) \setminus (A \times C)\). Then \((x, y) \in A \times B\), so \(x \in A\) and \(y \in B\). Also \((x, y) \notin A \times C\), so either \(x \notin A\) or \(y \notin C\). Since we have \(x \in A\), we get \(y \notin C\). Since \(y \in B\) it follows that \(y \in B \setminus C\). Hence \((x, y) \in A \times (B \setminus C)\). Therefore \((A \times B) \setminus (A \times C) \subseteq A \times (B \setminus C)\). We conclude that \(A \times (B \setminus C) = (A \times B) \setminus (A \times C)\).

(b)  We will assume that ii. and iii. are false and prove i. Since \(A \neq \emptyset\) there is some \(a\) such that \(a \in A\). Since \(B \neq \emptyset\) there is some \(b\) such that \(b \in B\). To prove \(A = B\) we use double containment. Suppose \(x \in A\). Then \((x, b) \in A \times B\). Since \(A \times B = B \times A\) we get \((x, b) \in B \times A\). Hence \(x \in B\). Therefore \(A \subseteq B\). Suppose \(y \in B\). Then \((a, y) \in A \times B\). Since \(A \times B = B \times A\) we get \((a, y) \in B \times A\). Hence \(y \in A\). Therefore \(B \subseteq A\). We conclude that \(A = B\).