THE ALEXANDER SUBBASE THEOREM AND THE TYCHONOFF THEOREM

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In this posting we give proofs of some theorems proved in class.

We say that $\mathcal{B}$ is a subbase for the topology of $X$ provided that (1) $B$ is open for every $B \in \mathcal{B}$ and (2) for every $x \in U \subset X$ with $U$ open, there is a finite collection $\{B_1, \ldots, B_n\} \subset \mathcal{B}$ such that

$$x \in \bigcap_{i=1}^{n} B_i \subset U.$$ 

**Theorem** (Alexander Subbase Theorem). Let $X$ be a topological space with a subbase $\mathcal{B}$. Suppose that for every cover $\mathcal{V} = \{B_a\}_{a \in A}$ of $X$ by elements of $\mathcal{B}$, there is a finite subcover $\{B_{a_1}, \ldots, B_{a_n}\}$. Then $X$ is compact.

**Proof.** Suppose not. Let $\mathcal{C}$ be an open cover of $X$ such that $\mathcal{C}$ has no finite subcover. We may assume that $\mathcal{C}$ is maximal with respect to this property. Note that $\mathcal{B} \cap \mathcal{C}$ cannot cover $X$. If it did, there would be a finite subcover by our assumption on $\mathcal{B}$. Let $x \in X \setminus \bigcup (\mathcal{B} \cap \mathcal{C})$. Now there is $U \in \mathcal{C}$ such that $x \in U$ and there are $\{B_1, \ldots, B_n\} \subset \mathcal{B}$ such that $x \in \bigcap_{i=1}^{n} B_i \subset U$. Clearly $B_i \not\in \mathcal{B} \cap \mathcal{C}$ by the choice of $x$. By the maximality of $\mathcal{C}$ we must have that $\mathcal{C} \cup \{B_i\}$ has a finite subcover for each $i$. Denote this $\{C_{1}^{i}, \ldots, C_{n_{i}}^{i}\} \cup \{B_i\}$. Then $\{C_{j}^{i}\}_{j=1}^{n_{i}} \cup \bigcap_{i=1}^{n} B_i$ is a finite open cover of $X$. This implies that $\{C_{j}^{n_{i}}\}_{j=1}^{n_{i}} \cup \{U\}$ is a finite open cover of $X$. However, this consists entirely of sets from $\mathcal{C}$, a contradiction. □

**Theorem** (The Tychonoff Theorem). Suppose that $\{X_{\alpha}\}_{\alpha \in A}$ is a collection of compact spaces. Then $\prod_{\alpha \in A} X_{\alpha}$ is compact.

**Proof.** By definition, a space $X$ is compact provided that for every open cover $\mathcal{U}$ of $X$, there is a finite sub cover. The Alexander Subbase Theorem says that this is equivalent to saying that if $\mathcal{B}$ is a subbase for $X$, then $X$ is compact if and only if every covering $\mathcal{U}$ of $X$ by elements from $\mathcal{B}$ has a finite subcovering.

Let $\mathcal{B} = \{\pi_{a}^{-1}(U) | U \text{ open } U \subset X_{\alpha} \text{ for some } \alpha\}$. This is a subbase for the topology of $\prod_{\alpha \in A} X_{\alpha}$. Let $\mathcal{U}$ be a covering of $X$ by elements of $\mathcal{B}$. Let $\mathcal{U}_{\alpha}$ be the subset of $\mathcal{U}$ which is associated with $U$ open in $X_{\alpha}$. Then we claim that there is an $\alpha_0$ such that $\mathcal{U}_{\alpha_0}$ covers $\prod_{\alpha \in A} X_{\alpha}$. If not, then for each $\alpha \in A$, let $z_\alpha$ be an element that is not covered by the any of the $U$ open in $X_{\alpha}$ such that $\pi^{-1}_{\alpha}(U) \in \mathcal{U}_{\alpha}$. Then clearly the point $(z_\alpha) \in \prod_{\alpha \in A} X_{\alpha}$ would not covered by $\mathcal{U}$. This is a contradiction.

So, let $\alpha_0$ be such that $\mathcal{U}_{\alpha_0}$ covers $\prod_{\alpha \in A} X_{\alpha}$. However, this implies that $X_{\alpha_0}$ is covered by the open sets $U \subset X_{\alpha_0}$ such that $\pi_{\alpha_0}^{-1}(U) \in \mathcal{U}_{\alpha_0}$. However, by assumption $X_{\alpha_0}$ is
compact. So, there are a finite number of these of these sets that cover $X_{\alpha_0}$. Call that collection $\{U_1, \ldots, U_n\}$. Then, $\{\pi_{\alpha_0}^{-1}(U_1), \ldots, \pi_{\alpha_0}^{-1}(U_n)\}$ is a finite cover of $\prod_{\alpha \in A} X_{\alpha}$ by elements of $U$. \qed