THE ALEXANDER SUBBASE THEOREM AND THE TYCHONOFF THEOREM

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In this posting we give proofs of some theorems proved in class.

We say that $\mathscr{B}$ is a *subbase for the topology* of X provided that (1) B is open for every $B \in \mathscr{B}$ and (2) for every $x \in U \subset X$ with U open, there is a finite collection $\{B_1, \ldots B_n\} \subset \mathscr{B}$ such that

$$x \in \bigcap_{i=1}^{n} B_i \subset U.$$

Theorem (Alexander Subbase Theorem)**.** *Let X be a topological space with a subbase $\mathscr{B}$. Suppose that for every cover $\mathscr{V} = \{B_a\}_{a \in A}$ of X by elements of $\mathscr{B}$, there is a finite subcover $\{B_{a_1}, \ldots, B_{a_n}\}$. Then X is compact.*

Proof. Suppose not. Let $\mathscr{C}$ be an open cover of X such that $\mathscr{C}$ has no finite subcover. We may assume that $\mathscr{C}$ is maximal with respect to this property. Note that $\mathscr{B} \cap \mathscr{C}$ cannot cover X. If it did, there would be a finite subcover by our assumption on $\mathscr{B}$. Let $x \in X \setminus \bigcup (\mathscr{B} \cap \mathscr{C})$. Now there is $U \in \mathscr{C}$ such that $x \in U$ and there are $\{B_1, \ldots, B_n\} \subset \mathscr{B}$ such that $x \in \bigcap_{i=1}^{n} B_i \subset U$. Clearly $B_i \notin \mathscr{B} \cap \mathscr{C}$ by the choice of x. By the maximality of $\mathscr{C}$ we must have that $\mathscr{C} \cup \{B_i\}$ has a finite subcover for each i. Denote this $\{C_1^i, \ldots, C_{n_i}^i\} \cup \{B_i\}$. Then $\{C_j^i\}_{j=1}^{n_i}{}_{i=1}^{n} \cup \{\bigcap_{i=1}^{n} B_i\}$ is a finite open cover of X. This implies that $\{C_j^i\}_{j=1}^{n_i}{}_{i=1}^{n} \cup \{U\}$ is a finite open cover of X. However, this consists entirely of sets from $\mathscr{C}$, a contradiction. $\square$

Theorem (The Tychonoff Theorem)**.** *Suppose that $\{X_\alpha\}_{\alpha \in A}$ is a collection of compact spaces. Then $\prod_{\alpha \in A} X_\alpha$ is compact.*

Proof. By definition, a space X is compact provided that for every open cover $\mathscr{U}$ of X, there is a finite sub cover. The Alexander Subbase Theorem says that this is equivalent to saying that if $\mathscr{B}$ is a subbase for X, then X is compact if and only if every covering $\mathscr{U}$ of X by elements from $\mathscr{B}$ has a finite subcovering.

Let $\mathscr{B} = \{\pi_\alpha^{-1}(U) \big| U \text{ open } U \subset X_\alpha \text{ for some } \alpha\}$. This is a subbase for the topology of $\prod_{\alpha \in A} X_\alpha$. Let $\mathscr{U}$ be a covering of X by elements of $\mathscr{B}$. Let $\mathscr{U}_\alpha$ be the subset of $\mathscr{U}$ which is associated with U open in X_α. Then we claim that there is an α_0 such that $\mathscr{U}_{\alpha_0}$ covers $\prod_{\alpha \in A} X_\alpha$. If not, then for each $\alpha \in A$, let z_α be an element that is not covered by the any of the U open in X_α such that $\pi_\alpha^{-1}(U) \in \mathscr{U}_\alpha$. Then clearly the point $(z_\alpha) \in \prod_{\alpha \in A} X_\alpha$ would not covered by $\mathscr{U}$. This is a contradiction.

So, let α_0 be such that $\mathscr{U}_{\alpha_0}$ covers $\prod_{\alpha \in A} X_\alpha$. However, this implies that X_{α_0} is covered by the open sets $U \subset X_{\alpha_0}$ such that $\pi_{\alpha_0}^{-1}(U) \in \mathscr{U}_{\alpha_0}$. However, by assumption X_{α_0} is compact. So, there are a finite number of these of these sets that cover X_{α_0}. Call that collection $\{U_1, \ldots, U_n\}$. Then, $\{\pi_{\alpha_0}^{-1}(U_1), \ldots, \pi_{\alpha_0}^{-1}(U_n)\}$ is a finite cover of $\prod_{\alpha \in A} X_\alpha$ by elements of $\mathscr{U}$. □