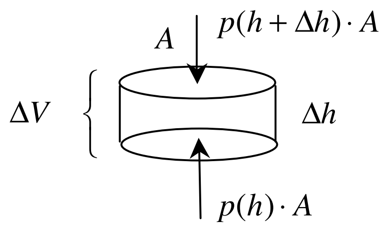
Atmospheric Pressure Decreases Exponentially

Suppose that $p(h)$ represents the atmospheric pressure at altitude h above the surface of the earth. We will show that this function has the form

$$p(h) = p(0)e^{-\frac{Mg}{RT}h}$$

where M is the gram molecular weight of the atmosphere, R is the universal gas constant, T is the absolute temperature, and g is the gravitational acceleration. Of course, these must all have values using consistent units.

Let ΔV be a small unit of volume which we visualize below. In the figure A is the cross-sectional area of the small volume and Δh is the height.



The force supporting this unit volume is the difference between the force due to the pressure on the bottom of the volume, given by $p(h)\cdot A$ and the force on the top of the volume given by $p(h+\Delta h)\cdot A$. This vertical force should just balance the weight of the gas in the volume which is given by $\Delta W = \rho(h)\cdot \Delta h \cdot A \cdot g$ where $\rho(h)$ is the density of the atmosphere at altitude h.

After some manipulation and taking a limit, we have the following differential equation.

$$\Delta W = \rho(h)\cdot A\cdot \Delta h\cdot g = p(h)\cdot A - p(h+\Delta h)\cdot A$$

$$\frac{p(h+\Delta h)\cdot A - p(h)\cdot A}{\Delta h} = -\rho(h)\cdot A\cdot g$$

$$\frac{dp}{dh} = -\rho(h)\cdot g$$

Using the ideal gas law, we can now related the pressure, $p(h)$, with the density, $\rho(h)$, of the atmosphere at altitude h. the ideal gas law is given below.

$$p\cdot V = n\cdot R\cdot T$$

Here p is the pressure, V is the volume of gas under consideration, n is the number of moles of the gas, and R and T are as stated above. If M is the gram molecular weight of the gas, then $n \cdot M$ is the weight. This leads to the equation

$$p = \left(\frac{n \cdot M}{V} \right) \frac{R \cdot T}{M} = \frac{R \cdot T}{M} \cdot \rho$$

Thus we have

$$\frac{dp}{dh} = -\frac{M \cdot g}{R \cdot T} \cdot p(h)$$

The solution of is differential equation is just

$$p(h) = p(0) e^{-\frac{Mg}{RT} h}$$

For standard units assuming that the gram molecular weight of the atmosphere is 28.8 g and standard temperature T

$$\frac{Mg}{RT} = 1.24426778 \cdot 10^{-4} \text{ per meter}$$

So, we get

$$p(h) = p(0) e^{-1.24426778 \cdot 10^{-4} h}$$

when we measure h in meters.