Bisection Method

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1 The Intermediate Value Theorem

The Bisection Method is a means of numerically approximating a solution to an equation.

$$f(x) = 0$$

The fundamental mathematical principle underlying the Bisection Method is the **Intermediate Value Theorem**.

Theorem 1.1. *Let $f : [a, b] \to [a, b]$ be a continuous function. Suppose that d is any value between $f(a)$ and $f(b)$. Then there is a c, $a < c < b$, such that $f(c) = d$.*

In particular, the Intermediate Value Theorem implies that if $f(a) \cdot f(b) < 0$, then there is a point c, $a < c < b$ such that $f(c) = 0$. Thus if we have a continuous function f on an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, then $f(x) = 0$ has a solution in that interval.

The Intermediate Value Theorem not only guarantees a solution to the equation, but it also provides a means of numerically approximating a solution to arbitrary accuracy.

Proceed as follows.

1. Let $\epsilon > 0$ be the upper bound for the error required of the answer.
2. Compute $c = \frac{a+b}{2}$ and $d = f(c) \cdot f(a)$
3. If $d < 0$, then let $b = c$ and $a = a$. If $d > 0$, then let $a = c$ and $b = b$. If $d = 0$, then c is a solution of $f(x) = 0$ and a solution has been found to the required accuracy.
4. The new interval $[a, b]$ will then be half the length of the original $[a, b]$ and will contain a point $x \in [a, b]$ such that $f(x) = 0$.

Repeat 2, 3, and 4 until either an exact solution is found in 3 or until at step 4 half the length of $[a, b]$ is less than ϵ, $\frac{b-a}{2} < \epsilon$.

2 Number of iterations

How many iterations are required for the solution to have the required accuracy? It can be easily seen that the number of steps n is given by the following formula.

$$\frac{b-a}{2^{n+1}} < \epsilon$$

$$n > \frac{\ln(b-a) - \ln(\epsilon)}{\ln(2)} - 1$$

It can also be easily seen that to reduce the error by a factor of 10 requires $\frac{\ln(10)}{\ln(2)} =$ 3.32192809887... more steps. This gives us the cost for each new digit of accuracy in the answer we obtain. Getting 100 digits more accuracy would thus require approximately 332 more steps.

Example 2.1. *Solve the equation* $x^{15} + 35 \cdot x^{10} - 20 \cdot x^3 + 10 = 0$.

There is a solution in the interval $[-3, 0]$ since $f(0) = 10$ and $f(-3) = -12,281,642$. The first few steps of the Bisection Method yield the following intervals.

1. $[-3, 0]$.
2. $[-3, -\frac{3}{2}] = [-3, -1.5]$ since $f(-\frac{3}{2}) > 0$.
3. $[-\frac{9}{4}, -\frac{3}{2}] = [-2.25, -1.5]$ since $f(-\frac{9}{4}) < 0$
4. $[-\frac{9}{4}, -\frac{15}{8}] = [-2.25, -1.875]$ since $f(-\frac{15}{8}) > 0$

   ...

A numerical solution is $x = -2.0378537990735054950...$ which is in the interval $[-2.25, -1.875]$. See the graph of the function on the next page. From the graph this seems to be the only zero in this interval.

3 Bisection Program for TI-89

Below is a program for the Bisection Method written for the TI-89. There are four input variables. The variable f is the function formula with the variable being x. In the case above, f would be entered as $x^{15} + 35 \cdot x^{10} - 20 \cdot x^3 + 10$. The variables a and b are the endpoints of the interval. It is assumed that $f(a) \cdot f(b) < 0$. The variable n is the number of iterations of the bisection method.

The matrix *bis* gives the endpoints of the intervals after each iteration beginning with the initial endpoints a and b.

```
:bisect(f,a,b,n)
:Prgm
:f → g
:NewMat(n+1,2) → bis
:approx(a) → a1
:approx(b) → b1
:a1 → bis[1,1]
:b1 → bis[1,2]
:For i,1,n,1
:(a1+b1)/2 → c
:If (g|x=a1)*(g|x=c)<0
:Then
:c → b1
:Else
:c → a1
:EndIf
:a1 → bis[i+1,1]
:b1 → bis[i+1,2]
:EndFor
:EndPrgm
```

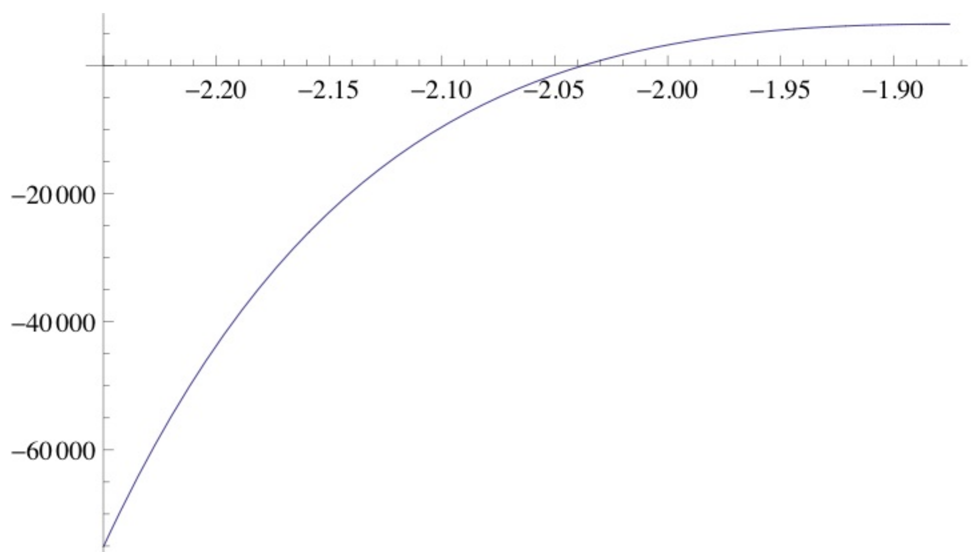

Figure 1: Graph of $x^{15} + 35 \cdot x^{10} - 20 \cdot x^3 + 10$ over $[-2.25, -1.875]$