THE CAUCHY MEAN VALUE THEOREM

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In this post we give a proof of the Cauchy Mean Value Theorem. It is a very simple proof and only assumes Rolle's Theorem.

Cauchy Mean Value Theorem Let $f(x)$ and $g(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then there is a $a < c < b$ such that

$$(f(b) - f(a)) \cdot g'(c) = (g(b) - g(a)) \cdot f'(c).$$

Proof. The case that $g(a) = g(b)$ is easy. So, assume that $g(a) \neq g(b)$. Define

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g(x).$$

Clearly, $h(a) = h(b)$. Applying Rolle's Theorem we have that there is a c with $a < c < b$ such that

$$h'(c) = 0 = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g'(c).$$

For this c we have that

$$(f(b) - f(a)) \cdot g'(c) = (g(b) - g(a)) \cdot f'(c).$$

□

The classical Mean Value Theorem is a special case of Cauchy's Mean Value Theorem. It is the case when $g(x) \equiv x$. The Cauchy Mean Value Theorem can be used to prove **L'Hospital's Theorem**.