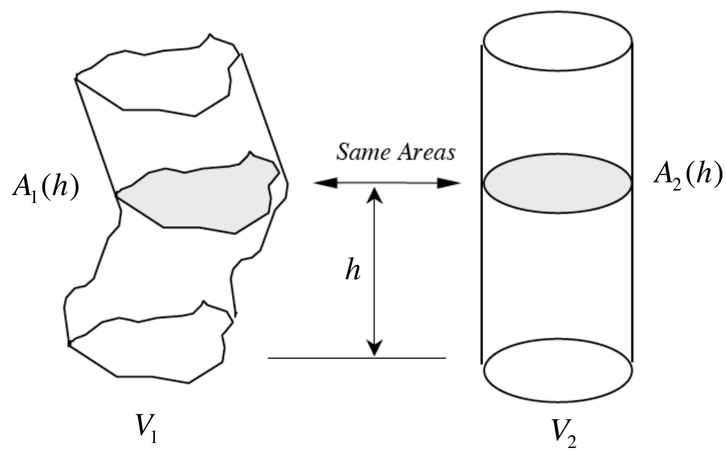
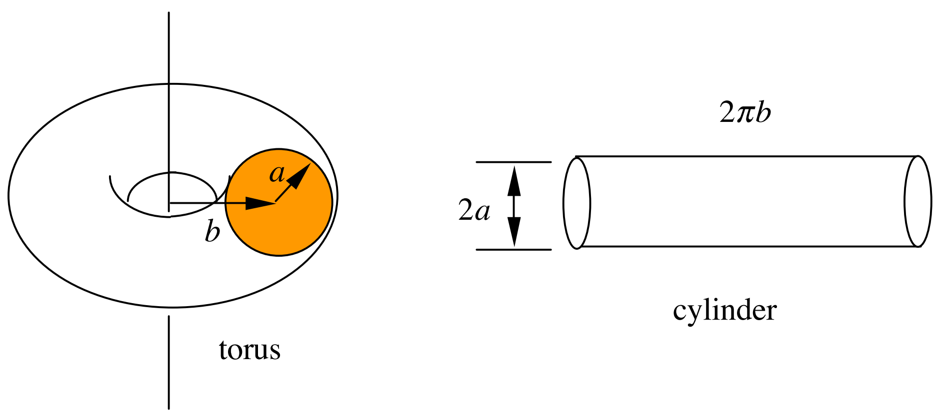
Cavalieri's Determination of the Volume of a Torus

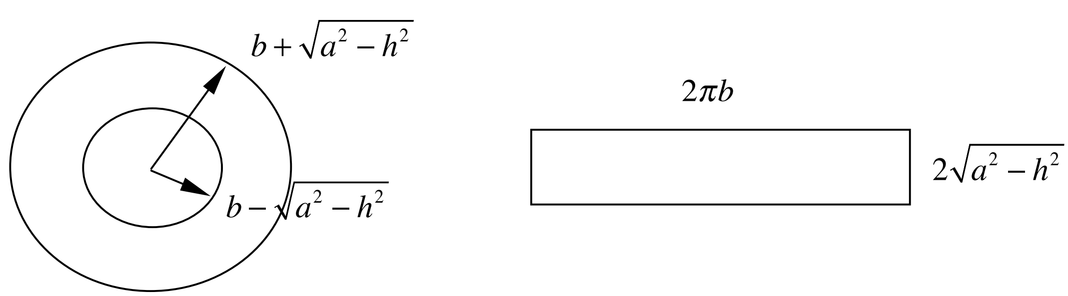
Bonaventura Cavalieri (1598-1647) was a contemporary of Galileo who considered him the greatest geometer since Archimedes. One of his powerful tools has been called *Cavalieri's Principle*. It states that if two solids have the same height and at each level have the same cross sectional area, then they both have the same volume. This is represented with a mnemonic diagram below.



The volume of a torus can be determined in this way by comparing its cross-sectional areas with that of a cylinder. The comparison is below.



At a height *h* from the center of the figures we get the following cross sections.



So, the area for the torus is $A_1(h) = \pi\left(b + \sqrt{a^2 - h^2}\right)^2 - \pi\left(b - \sqrt{a^2 - h^2}\right)^2 = 4\pi b\sqrt{a^2 - h^2}$.

The area for the cylinder is clearly $A_2(h) = 4\pi b\sqrt{a^2 - h^2}$. So, $A_1(h) = A_2(h)$ for every h. This implies that the volumes are the same. So, the volume of a torus is given by the following.

$$V = 4\pi^2 ba^2$$