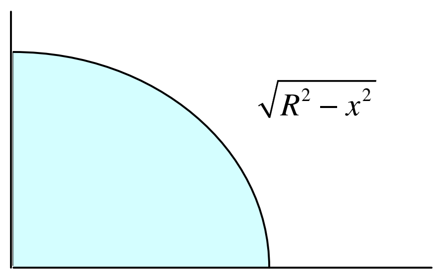
Centroids

Let A be a geometric figure in Euclidean space. The Euclidean space may be R^2 or R^3. It could in fact be R^n. The figure may be any dimension. The coordinates of the *centroid* or *center of mass* of the figure are given by the following formulas.

$$\overline{x} = \frac{\int x \cdot dA}{\int dA} \quad \overline{y} = \frac{\int y \cdot dA}{\int dA} \quad \overline{z} = \frac{\int z \cdot dA}{\int dA}$$

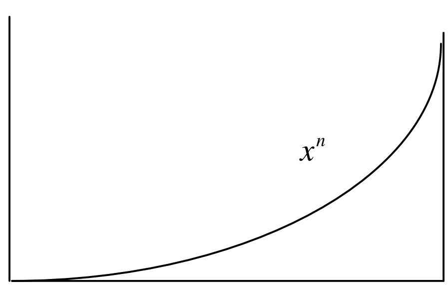
Of course, if the figure is in R^2, only the first two coordinates apply. The dA in the formulas are small strips of the figure that are perpendicular to the direction of the respective *x*–axis or *y*–axis or *z*–axis. The simplest way to understand this is to work through a few examples.

1. Determine the centroid of the portion of the area of a disk of radius R centered at the origin that is in the first quadrant.



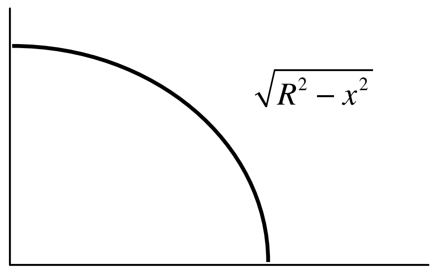
$$\overline{x} = \frac{\int_0^R x \cdot \sqrt{R^2 - x^2}\, dx}{\int_0^R \sqrt{R^2 - x^2}\, dx} = \frac{4R}{3\pi} \quad \overline{y} = \frac{\int_0^R y \cdot \sqrt{R^2 - y^2}\, dx}{\int_0^R \sqrt{R^2 - y^2}\, dx} = \frac{4R}{3\pi}$$

2. Determine the centroid of the area under the curve $f(x) = x^n$ over the interval $[0,1]$.



$$\overline{x} = \frac{\int_0^1 x \cdot x^n\, dx}{\int_0^1 x^n\, dx} = \frac{n+1}{n+2} \quad \overline{y} = \frac{\int_0^1 y \cdot (1 - \sqrt[n]{y})\, dy}{\int_0^1 (1 - \sqrt[n]{y})\, dy} = \frac{n+1}{4n+2}$$

3. What is the centroid of that portion of the graph of a circle centered at the origin having radius R that is in the first quadrant.



$$\bar{x} = \frac{\int_0^R x \cdot \sqrt{1 + \left(\frac{d}{dx}\sqrt{R^2 - x^2}\right)^2}\, dx}{\int_0^R \sqrt{1 + \left(\frac{d}{dx}\sqrt{R^2 - x^2}\right)^2}\, dx} = \frac{2R}{\pi} \qquad \bar{y} = \frac{\int_0^R y \cdot \sqrt{1 + \left(\frac{d}{dx}\sqrt{R^2 - y^2}\right)^2}\, dy}{\int_0^R \sqrt{1 + \left(\frac{d}{dx}\sqrt{R^2 - y^2}\right)^2}\, dy} = \frac{2R}{\pi}$$

One can easily derive Pappus' Rule from the formula for the centroid. Suppose that the figure is to be rotated about the y–axis and the resultant area or volume is to be determined. Then, in the case of volume, the formula is given below.

$$V = 2 \cdot \pi \cdot \bar{x} \cdot A$$

The area of the figure is A and the x–coordinate of the centroid is $\bar{x}$.

Here is the proof using the definition of $\bar{x}$.

$$V = \int 2 \cdot \pi \cdot x \cdot dA = 2 \cdot \pi \int x \cdot dA = \frac{2 \cdot \pi \int x \cdot dA \int dA}{\int dA} = 2 \cdot \pi \cdot \bar{x} \cdot A$$

Let us apply Pappus' Rule to Example 1 above. If we rotate the area in the quarter disk around the x–axis, we get a half sphere. The volume is $\frac{2}{3}\pi \cdot R^3$. The area of the quarter disk is $\frac{\pi \cdot R^2}{4}$. From these two facts, we can determine $\bar{y}$.

$$V = \frac{2}{3}\pi \cdot R^3 = 2 \cdot \pi \cdot A = 2 \cdot \pi \cdot \bar{y} \cdot \frac{\pi \cdot R^2}{4}$$

It is a simple calculation to see that $\bar{y} = \frac{4R}{3\pi}$, the same answer that was obtained by integrating.

We can apply Pappus' Rule to determine the volume obtained by rotating the area in Example 2 around the *y*–axis.

$$V = 2 \cdot \pi \cdot \bar{x} \cdot A = 2 \cdot \pi \cdot \frac{n+1}{n+2} \cdot \frac{1}{n+1} = \frac{2 \cdot \pi}{n+2}$$

If we would rotate around the *x*–axis, the volume we would the following.

$$V = 2 \cdot \pi \cdot \bar{y} \cdot A = 2 \cdot \pi \cdot \frac{n+1}{4n+2} \cdot \frac{1}{n+1} = \frac{\pi}{2n+1}$$