# ***Contraction Mapping Theorem***

Below is a statement and proof of the Contraction Mapping Theorem.

**Theorem.** *Suppose that X is a complete metric space and that $f: X \to X$ is a contraction mapping on X. Then there is a unique $z \in X$ such that $f(z) = z$. Furthermore, if $x_0$ is any point in X, then $f^n(x_0) \to z$ as $n \to \infty$.*

*Proof.* Let $x_0 \in X$. We will show that the sequence $\left\{x_n = f^n(x_0)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in *X*. Let $D = d(x_0, f(x_0))$ and let $0 < c < 1$ be the contraction constant for *f*. By the definition of contraction mapping $d(f(x_0), f^2(x_0)) \le c \cdot d(x_0, f(x_0)) = c \cdot D$. By induction one can establish that $d(f^n(x_0), f^{n+1}(x_0)) \le c^n \cdot D$. Thus,

$$d(x_0, f^{n+1}(x_0)) \le \sum_{k=0}^{n} c^k \cdot D < \frac{D}{1-c}$$

Similarly, $d(f^n(x_0), f^m(x_0)) \le D \cdot c^n \cdot \sum_{k=0}^{m-n-1} c^k < \frac{c^n \cdot D}{1-c}$ for all $n < m$. So, to see that the sequence is Cauchy, let $\varepsilon > 0$ and choose *N* such that $\frac{c^N \cdot D}{1-c} < \varepsilon$. Then for $N \le n \le m$, $d(f^n(x_0), f^m(x_0)) < \varepsilon$. So, the sequence is Cauchy.

Since $\left\{x_n = f^n(x_0)\right\}_{n=1}^{\infty}$ is Cauchy, it converges to a point $z \in X$. But for this *z* $\lim_{n\to\infty} f^n(x_0) = z = \lim_{n\to\infty} f^{n+1}(x_0) = f(z)$. So, *z* is a fixed point for *f*. On the other hand, if there were another fixed point $z' \ne z$, then $d(f(z), f(z')) = d(z, z') > c \cdot d(z, z')$. This last inequality contradicts the assumption that *f* is a contraction mapping. So, there is only one fixed point.