**Contraction Mapping Theorem**

Below is a statement and proof of the Contraction Mapping Theorem.

**Theorem.** Suppose that \( X \) is a complete metric space and that \( f : X \to X \) is a contraction mapping on \( X \). Then there is a unique \( z \in X \) such that \( f(z) = z \).

Furthermore, if \( x_0 \) is any point in \( X \), then \( f^n(x_0) \to z \) as \( n \to \infty \).

**Proof.** Let \( x_0 \in X \). We will show that the sequence \( \{x_n = f^n(x_0)\}_{n=1}^\infty \) is a Cauchy sequence in \( X \). Let \( D = d(x_0, f(x_0)) \) and let \( 0 < c < 1 \) be the contraction constant for \( f \).

By the definition of contraction mapping \( d(f(x_0), f^2(x_0)) \leq c \cdot d(x_0, f(x_0)) = c \cdot D \). By induction one can establish that \( d(f^n(x_0), f^{n+1}(x_0)) \leq c^n \cdot D \). Thus,

\[
d(x_0, f^{n+1}(x_0)) \leq \sum_{k=0}^{n} c^k \cdot D < \frac{D}{1-c}
\]

Similarly, \( d(f^n(x_0), f^m(x_0)) \leq D \cdot c^n \cdot \sum_{k=0}^{m-n-1} c^k < \frac{c^n \cdot D}{1-c} \) for all \( n < m \). So, to see that the sequence is Cauchy, let \( \varepsilon > 0 \) and choose \( N \) such that \( \frac{c^N \cdot D}{1-c} < \varepsilon \). Then for \( N \leq n \leq m \),

\[
d(f^n(x_0), f^m(x_0)) < \varepsilon \quad \text{. So, the sequence is Cauchy.}
\]

Since \( \{x_n = f^n(x_0)\}_{n=1}^\infty \) is Cauchy, it converges to a point \( z \in X \). But for this \( z \)

\[
\lim_{n \to \infty} f^n(x_0) = z = \lim_{n \to \infty} f^{n+1}(x_0) = f(z) \quad \text{. So, } z \text{ is a fixed point for } f \text{.}
\]

On the other hand, if there were another fixed point \( z' \neq z \), then \( d(f(z), f(z')) = d(z, z') > c \cdot d(z, z') \). This last inequality contradicts the assumption that \( f \) is a contraction mapping. So, there is only one fixed point.