THE CANTOR SET AND A CONTINUOUS MAP FROM THE INTERVAL ONTO THE SQUARE

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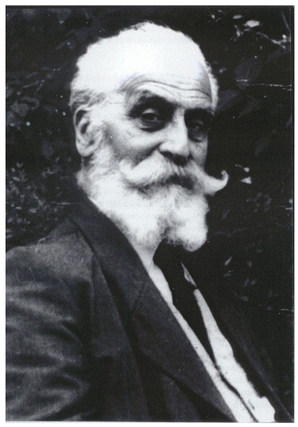

Figure 1. Giuseppe Peano, 1858-1932

Figure 2. David Hilbert, 1862-1943

In this post describe and discuss the Cantor Set, $C \subset I$. We use this set to construct a continuous map $f : [0,1] \to [0,1] \times [0,1]$ which is onto. The steps in the proof are not difficult, but the final result is surprising. The first such map was discovered by Giuseppe

Peano in 1890. There was another example published by David Hilbert in 1891. The examples were a shock to the mathematical community at the time and demonstrated that the notion of dimension needed a logical foundation.

Let us state at the outset the main result that we are concerned with.

Theorem: Dimension Raising Mapping. There is a continuous mapping $f : [0,1] \to [0,1] \times [0,1]$ which is onto.

The proof of this result will come at the end of the document.

1. The Cantor Set

We first introduce the Cantor Set. It is the set of points that can be expressed in a ternary expansion using only the digits 0 and 2. It is formally given by the following.

$$C = \left\{ x \in I = [0,1] \,\middle|\, x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \;\text{ with}\, a_i \in \{0,2\} \,\text{for all}\; i \right\}$$

or

$$C = \{ x \in I = [0,1] \,|\, x = .a_1\, a_2\, a_3\, a_4 \cdots {}_3 \;\text{ with}\, a_i \in \{0,2\} \,\text{for all}\; i \}$$

One can visualize this set in the following way. Let $A_0 = [0,1]$. Let $A_1 \subset A_0$ be $[0,1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right)$. Note that A_1 is the set of points that can be represented with either 0 or 2 as the first digit. Let A_2 be the two intervals in A_1 less their middle thirds. Then A_2 is the set of numbers in $[0,1]$ that can be represented in ternary expansion with 0 or 2 in the first two positions. Continuing in this fashion we see that $C = \cap_{n=1}^{\infty} A_n$. This construction is the reason that it is often known as the Cantor Middle-Third Set.

Figure 3. The Cantor Middle-Third Set

We observe that some elements of $[0,1] = I$ have two ternary expansions. If one of them uses only the digits 0 and 2, then that is the expansion that is used. For example $\frac{1}{3} = .1\,0\,0 \cdots {}_3 = .0\,2\,2\,2 \cdots {}_3$. So, in this case we use the second expansion. For $\frac{2}{3}$ we use the expansion $.2\,0\,0\,0 \cdots {}_3$ even though $.1\,2\,2\,2 \cdots {}_3$ is another.

2. Continuous Maps on the Cantor Set

Theorem: Continuous Map of C onto I. There is a continuous mapping $f : C \to I$ which is onto.

Proof. For $x \in C$ let $x_0 = .a_1\, a_2\, a_3 \cdots {}_3$ with $a_i \in \{0, 2\}$ Note that $\frac{a_i}{2} \in \{0, 1\}$ for all i. So, if we define $f(x) = y$ where $y = .\frac{a_1}{2}\, \frac{a_2}{2}\, \frac{a_3}{2} \cdots {}_2$ we will get a number $y \in I$ with the given binary expansion. If we only use the agreed upon ternary expansions in C that use the digits 0 and 2, there will be only one such y for each $x \in C$. It is not difficult to show that this function is continuous. We now show that it is onto. Let y be any element of I. Let y have the binary expansion $y = .b_1\, b_2\, b_3 \cdots {}_2$ with $b_i \in \{0, 1\}$ for all i. Let $x \in I$ have the ternary expansion $x = .2b_1\, 2b_2\, 2b_3\, 2b_4 \cdots {}_3$. Then clearly $x \in C$ and $f(x) = y$. So, f is onto I. □

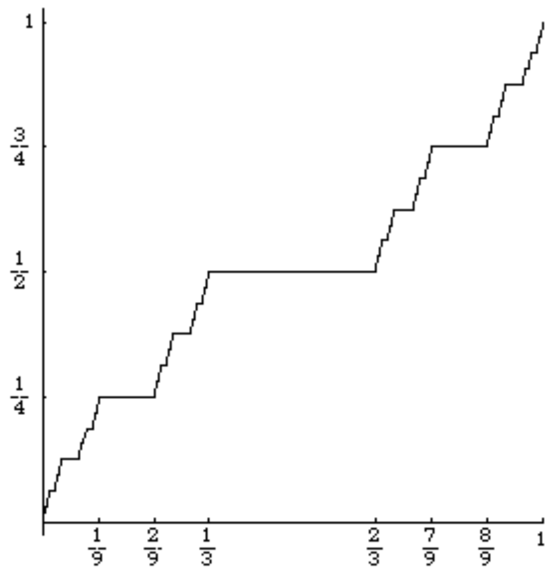

Figure 4. The Graph of g is the Devil's Staircase

We observe that $C = \cap_{i=n}^{\infty} A_n$ as we described earlier where A_n is a union of 2^n intervals each of length 3^{-n}. We can extend our map $f : C \to I$ to a map $g : I \to I$ through the following observation. Each point in I will either be in all of the A_n's or will be in one of the complementary intervals for some A_n. So, we will start with the map $f : C \to I$ that we have constructed and extend that map to $g : I \to I$ by defining $g(x)$ to be constant on each of the complementary intervals of A_n for each n. This is possible since $f\left(\frac{1}{3}\right) = f\left(\frac{2}{3}\right) = \frac{1}{2}$, $f\left(\frac{1}{9}\right) = f\left(\frac{2}{9}\right) = \frac{1}{4}$, $f\left(\frac{7}{9}\right) = f\left(\frac{8}{9}\right) = \frac{3}{4}$, etc. The graph in Figure 4 helps visualize the function g. It is known as the Devil's Staircase.

The map $f : C \to I$ is really an introduction to a continuous map $f : C \to I \times I$ which is onto. We now show that such a map exists. The description is similar to that of $f : C \to I$ just described.

Theorem: Continuous Map of C onto $I \times I$. There is a continuous mapping $f : C \to I \times I$ which is onto.

Proof. For $x \in C$ let $x_0 = .a_1\, a_2\, a_3\, \cdots\, _3$ with $a_{2i-1} \in \{0, 2\}$ Note that $\frac{a_{2i-1}}{2} \in \{0, 1\}$ for all i. So, if we define $f(x) = (y_1, y_2)$ where we let $y_1 = .\frac{a_1}{2}\, \frac{a_3}{2}\, \frac{a_5}{2}\, \cdots\, _2$ and $y_2 = .\frac{a_2}{2}\, \frac{a_4}{2}\, \frac{a_6}{2}\, \cdots\, _2$. In this way we will get a point $(y_1, y_2) \in I \times I$.

If we only use the agreed upon ternary expansions in C that use the digits 0 and 2, there will be only one such (y_1, y_2) for each $x \in C$. So, the function will be *well defined*. It is not difficult to show that this function is continuous. We now show that it is onto. Let (y_1, y_2) be any element of $I \times I$. Let y_1 have the binary expansion $y_1 = .b_1\, b_2\, b_3\, \cdots\, _2$ with $b_i \in \{0, 1\}$ for all i. Let y_2 have the binary expansion $y_2 = c_1\, c_2\, c_3\, \cdots\, _2$. Let $x \in I$ have the ternary expansion $x = .2b_1\, 2c_1\, 2b_2\, 2c_2\, \cdots\, _3$. Then clearly $x \in C$ and $f(x) = (y_1, y_2) \in I \times I$ the way we have defined it. So, f is onto $I \times I$. $\square$

Can you come up with a way to describe a continuous function $f : C \to I^3$ or onto a higher dimensional cube? What about an infinite-dimensional cube?

This does not give us the continuous map from I onto I^2 that was promised. We now provide a proof of that claim. We observe that $C = \cap_{i=n}^{\infty} A_n$ as we described earlier where A_n is a union of 2^n intervals each of length 3^{-n}. Our strategy will be to extend the map $f : C \to I^2$ from the previous theorem to the intervals that are complementary to A_n for each n. Each point in I will either be in all of the A_n's or will be in one of the complementary intervals for A_n for some n. So, we will start with the map $f : C \to I^2$ that we have constructed and extend that map to $\overline{f} : I \to I^2$ by defining $\overline{f}$ on each of these complementary intervals.

3. Proof of the Main Theorem

Proof. Let $f : C \to I^2$ as described in the previous theorem. Let (a, b) be a complementary interval for A_n. Then $a = \frac{i}{3^n}$ for some positive integer i and $b = \frac{i+1}{3^n}$ for the same i. Note that the distance between $f(a)$ and $f(b)$ is no farther apart than $\frac{1}{2^{2\cdot(n-1)}}$. Now define $\overline{f}$ on the interval $[a, b]$. Let $x \in [a, b]$. Then x is given by $x = a + t_x \cdot (b - a)$ for a unique $t_x \in [0, 1]$. Now define $\overline{f}(x)$ by the formula $\overline{f}(a + t_x \cdot (b - a) = f(a) + t_x \cdot (f(b) - f(a)$. We have defined $\overline{f}$ on $[a, b]$ linearly onto the line joining $f(a)$ and $f(b)$ in I^2. The map $\overline{f}$ will be onto since f was onto. With some work this map can also be seen to be continuous. $\square$

It helps to visualize the lines that the complementary intervals map to. Note that $f\left(\frac{1}{3}\right) = f(.0\,2\,2\,2\,\cdots\,_3) = \left(\frac{1}{2}, 1\right)$ and $f\left(\frac{2}{3}\right) = f(.2\,0\,0\,0\,\cdots\,_3) = \left(\frac{1}{2}, 0\right)$. So, $\overline{f}\left(\left[\frac{1}{3}, \frac{2}{3}\right]\right)$ is the vertical line joining $\left(\frac{1}{2}, 1\right)$ to $\left(\frac{1}{2}, 0\right)$ directed downward. The images of the other complementary intervals can be determined in a similar fashion. Figure 5 helps to visualize the image of the first few complementary intervals in the square. The arrowhead indicates the direction of the image of the respective interval.

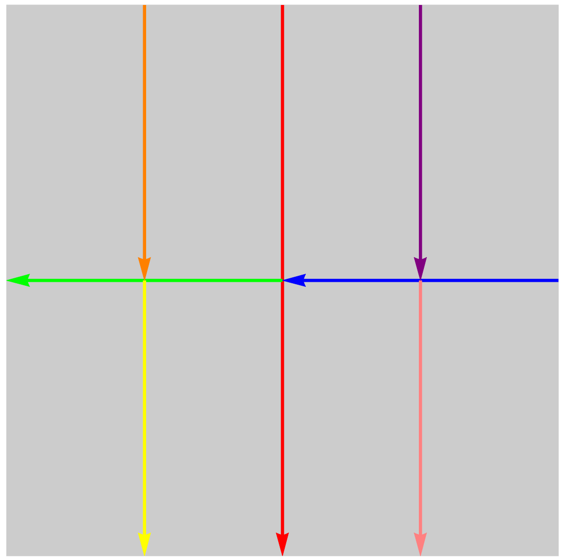

FIGURE 5. The Image under $\overline{f}$ of $\left[\frac{1}{3}, \frac{2}{3}\right]$ (Red), $\left[\frac{1}{9}, \frac{2}{9}\right]$ (Green), $\left[\frac{7}{9}, \frac{8}{9}\right]$ (Blue), $\left[\frac{1}{27}, \frac{2}{27}\right]$ (Yellow), $\left[\frac{7}{27}, \frac{8}{27}\right]$ (Orange), $\left[\frac{19}{27}, \frac{20}{27}\right]$ (Pink), $\left[\frac{25}{27}, \frac{26}{27}\right]$ (Purple) in $I \times I$