EXACT DIFFERENTIAL EQUATIONS

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In this post we give the basic theory of exact differential equations. These equations arise from a function of the form

$$F(x, y) = C$$

where C is a constant.

Such an equation can be converted to a differential equation in the following manner.

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

In this manner we have a first-order differential equation. The solutions of this equation are curves $y(x)$ such that $F(x, y(x)) \equiv C$ for some constant C.

It is sometimes more useful to write the differential equation in the following way.

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

This leads to a general family of differential equations of the following form.

$$M(x, y)dx + N(x, y)dy = 0 \tag{1}$$

The question arises, When does the equation above come from a problem

$$F(x, y) = C?$$

This is equivalent to asking, When is there a function $F(x, y)$ such that $\frac{\partial F}{\partial x} = M(x, y)$ and $\frac{\partial F}{\partial y} = N(x, y)$? When this is true, then the following holds.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2}$$

This is because

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

If we have that equations (1) and (2) hold, then we can readily solve the differential equation.

1. Example

Here is an example. Suppose that the following is a particular case of (1).

$$x\,dx + y\,dy = 0 \tag{3}$$

We note that $\frac{\partial x}{\partial y} = 0 = \frac{\partial y}{\partial x}$ so that (2) holds in this case. Now we can determine the equation $F(x, y) = C$ that gives rise to (1).

$$\int x\,\partial x = \frac{x^2}{2} + g(y)$$

for some arbitrary function $g(y)$.
Similarly,

$$\int y\,\partial y = \frac{y^2}{2} + h(x)$$

for some arbitrary function $h(x)$. This gives us

$$F(x, y) = \frac{x^2}{2} + g(y) = \frac{y^2}{2} + h(x)$$

which implies that $g(y) = \frac{y^2}{2} + C$ and $h(x) = \frac{x^2}{2} + C$ where C is some arbitrary constant. Thus, $x^2 + y^2 = C$ gives the family of solutions of (3).

2. General Solution

In fact, whenever we have a differential equation of the form (1) for which (2) holds, then there is a solution to the equation of the form $F(x, y) = C$. The function $F(x, y)$ can be found by the procedure in the example above. Specifically,

$$\int M(x, y)\,\partial x + h(y) = \int N(x, y)\,\partial y + g(x)$$

and then solve for $h(y)$ and $g(x)$. This will always be possible if (2) holds.

3. Integrating Factors

Suppose that we have an equation of the form (1) which is not exact. Note that we could write the same equation as

$$\mu(x, y) \cdot M(x, y)\,dx + \mu(x, y) \cdot N(x, y)\,dy = \mu(x, y) \cdot (M(x, y)\,dx + N(x, y)\,dy) = 0 \tag{4}$$

which would be equivalent whenever $\mu(x,y) \neq 0$. It may be possible to multiply (1) by such a function $\mu(x,y)$ so that the new equation (4) is exact. In that case we would say that $\mu(x,y)$ is an *integrating factor* for (1).

Integrating factors are not so easy to come by. However, they exist in a few cases that are good to know. We will only cover three cases. The first case is obtained by supposing the we have $\mu(x,y) \equiv \mu(x)$. That is, our function μ is a function of x alone. Assuming that we get the following equation.

$$\frac{d\mu}{dx} = \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) \cdot \mu \tag{5}$$

If the right hand side of (5) is a function of x alone, then we can determine $\mu(x)$ by integrating. The function $\mu(x)$ is given by the following.

$$\mu(x) = \exp\left[\int \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) dx \right]$$

Another case is similar. Suppose that $\mu(x,y) = \mu(y)$ is simply a function of y alone. In a similar fashion we solve for $\mu(y)$ to get the following

$$\mu(y) = \exp\left[\int \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \right) dy \right]$$

provided that

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

is a function of y alone. In these two cases we multiply original equation by the integrating factor and solve the resulting equation as an exact equation.

4. Example

Consider the following equation.

$$x^2\,dx + xy\,dy = 0 \tag{6}$$

Note that this equation is not exact since $\frac{\partial(x^2)}{\partial y} = \frac{M(x,y)}{\partial y} \equiv 0$ which is not equal to $\frac{\partial N(x,y)}{\partial x} = \frac{\partial(xy)}{\partial x} \equiv y$

However,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-y}{x \cdot y} = \frac{-1}{x}$$

is a function of x alone. Therefore,

$$\exp\left[\int \left(\frac{-1}{x}\right) dx\right] = \frac{1}{x}$$

is an integrating factor for the equation. When we multiply (6) by $\frac{1}{x}$ we get equation (3) which we have already solved. So, (6) has the same solution as (3).

The third type of equation that can be solved using an integrating factor is the first-order linear differential equation. We will cover that case in a separate posting.