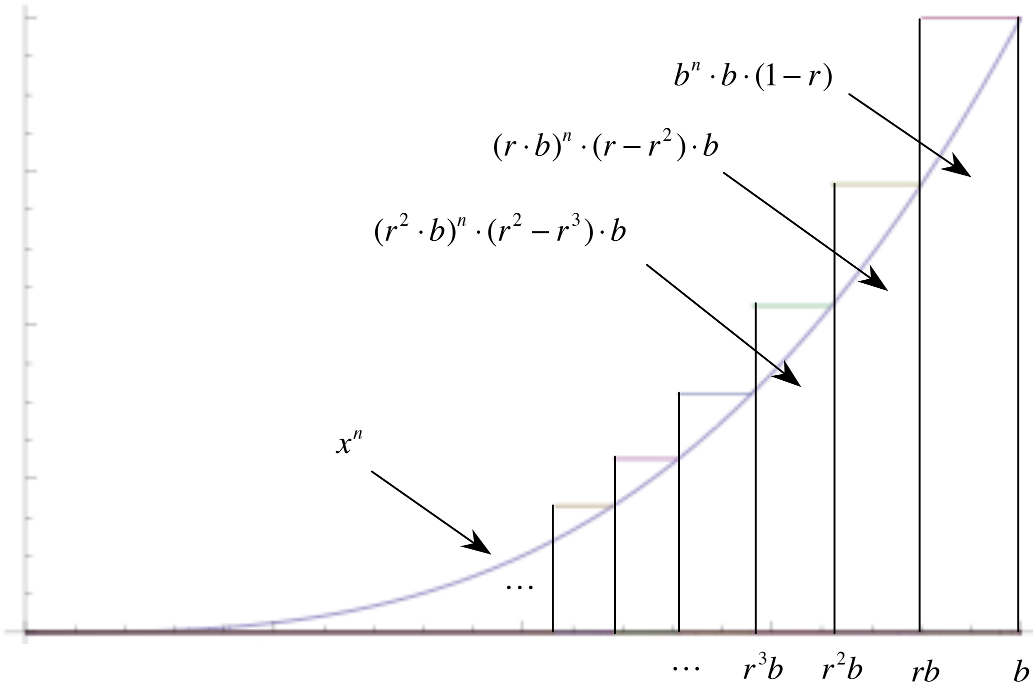
Fermat's Determination of the Area under x^n over [0,b]

Consider the function $f(x) = x^n$ over the interval $[0,b]$. Let r be any number such that $0 < r < 1$. Fermat got the idea to approximate the area under the curve with an infinite number of rectangles as in the graph below. Only the rightmost six rectangles are shown on the graph. The areas of the rightmost three of the rectangles are .



The area of the k^{th} rectangle can be seen to have the following formula.

$$(r^k \cdot b)^n \cdot r^k \cdot b \cdot (1-r) = b^{n+1} \cdot (1-r) \cdot r^{(n+1)\cdot k}$$

So the total area of the rectangles is given by the following.

$$b^{n+1} \cdot (1-r) \cdot \sum_{k=0}^{\infty} r^{(n+1)\cdot k}$$

Using the formula for the Geometric Series we get the following.

$$\frac{b^{n+1} \cdot (1-r)}{1 - r^{n+1}} = \frac{b^{n+1}}{1 - r^{n+1} \big/ 1 - r} = \frac{b^{n+1}}{1 + r + r^2 + \cdots + r^n}$$

Then Fermat took the limit of this expression as $r \to 1$ from below.

$$\lim_{r \nearrow 1} \frac{b^{n+1}}{1+r+r^2+\cdots+r^n} = \frac{b^{n+1}}{n+1}$$

This is an example of the insight that was necessary to determine the area under a curve for each new curve that was considered. It was necessary to see the curve in some geometric way that allowed the insight to determine its area. Often a special calculation was also required.

Calculus made it possible to determine areas by finding antiderivatives. There was only *one* trick involved each time.

[1] G. Simmons, *Calculus Gems*, McGraw-Hill, 1992, p. 240.