Gaussian Quadrature

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1 Quadrature Using Points with Unequal Spacing

In Newton-Cotes Integration we used points that were equally spaced. However, there was no need for the points to have any special spacing. If we wish to estimate the integral

$$\int_a^b f(x)\,dx$$

and have any set of points $\{x_0, x_1, \ldots, x_n\}$, then we can estimate the integral by the formula

$$\int_a^b f(x)\,dx \approx A_0 \cdot f(x_0) + A_1 \cdot f(x_1) + \cdots + A_n \cdot f(x_n).$$

We can solve for the constants $\{A_0, A_1, \ldots, A_n\}$ by making the formula exact for the functions $f(x) = 1, x, x^2, x^3, \ldots, x^n$. This will give us $n+1$ equations that we can use to solve for the constants $\{A_0, A_1, A_2, \ldots, A_n\}$.

In Gaussian Quadrature we use the interval $[-1, 1]$ as the standard and the points $\{x_0, x_1, \ldots, x_n\}$ will all be contained in this interval. There is a matrix equation for the normalized constants $\{a_0, a_1, \ldots, a_n\}$.

$$M = \text{Vandermonde}([x_0, x_1, \ldots, x_n]) = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

Let M^T be the transpose of M. Let A be the column vector with entries a_i. Let B be a column vector with entries

$$b_i = \int_{-1}^{1} x^i\,dx = \frac{1 - (-1)^{i+1}}{i+1}.$$

Then we get

$$M^T \cdot A = B$$

and solving for A we get the following.

$$A = \left(M^T\right)^{-1} \cdot B$$

2 Choosing the Points

We now have flexibility to choose the points $\{x_0, x_1, \ldots, x_n\}$ in a way that will make the estimate of the integral even more accurate. The theory behind the choice of points involves the Legendre polynomials. Let us denote these polynomials by $\{P_n(x) \mid n = 0, 1, 2, \ldots\}$. These polynomials have the following properties.

(1) $P_n(x)$ has degree n.
(2) If $i \neq j$, then

$$\int_{-1}^{1} P_i(x) \cdot P_j(x)\, dx = 0.$$

(3) The vector span of $\{P_0(x), P_1(x), \ldots, P_n(x)\}$ is the same as that of $\{1, x, x^2, \ldots, x^n\}$.
(4) For each n

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}.$$

Property (3) implies that $\int_{-1}^{1} P_n(x) \cdot P(x)\, dx = 0$ for any polynomial $P(x)$ of degree less than n.

Here are the first few Legendre polynomials.

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) \\
P_6(x) &= \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) \\
P_6(x) &= \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x) \\
&\vdots
\end{aligned}$$

There is a simple formula that gives these polynomials.

$$P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n}\left((x^2-1)^n\right)$$

The roots of these polynomials are all real and distinct. They are contained in the interval $[-1, 1]$. The points used in Gaussian Quadrature are the roots of P_{n+1}, $\{x_0, x_1, \ldots, x_n\}$. Because of the properties of the Legendre polynomials, it turns out that if $P(x)$ is any polynomial of degree k up to $2n + 1$, then the Gaussian Quadrature estimate of the integral of $P(x)$ is exact. We now prove this fact.

Theorem 2.1. *Suppose that $\{x_0, x_1, x_2, \ldots, x_n\}$ are the roots of the Legendre polynomial of degree $n + 1$. Suppose that $\{a_0, a_1, \ldots, a_n\}$ are the normalized coefficients for Gaussian Quadrature for these points. Then for any interval $[a, b]$,*

$$\int_a^b P(x)\,dx = \frac{b-a}{2} \sum_{i=0}^{n} a_i \cdot f\left(\frac{b-a}{2} \cdot x_i + \frac{a+b}{2}\right).$$

Proof. We will give the proof just for the interval $[-1, 1]$. Adapting the proof to a general interval just requires a careful change of variables. Let $P_{n+1}(x)$ be the Legendre polynomial with its roots $\{x_0, x_1, \ldots, x_n\}$. Assume that $P(x)$ has degree $\leq 2n + 1$. Let $P(x) = Q(x) \cdot P_{n+1}(x) + R(x)$ be the quotient and remainder. Then the degree of $Q(x)$ is $\leq n$ and the degree of $R(x)$ also has degree $\leq n$. Consider the integral

$$\int_{-1}^1 P(x)\,dx = \int_{-1}^1 Q(x) \cdot P_{n+1}(x) + R(x)\,dx.$$

Since the degree of $Q(x)$ is less than that of $P_{n+1}(x)$ we will have

$$\int_{-1}^1 Q(x) \cdot P_{n+1}(x)\,dx = 0.$$

So, we get the following.

$$\begin{aligned}\int_{-1}^1 P(x)\,dx &= \int_{-1}^1 P_{n+1}(x) \cdot Q(x) + R(x)\,dx \\ &= \int_{-1}^1 R(x)\,dx\end{aligned}$$

On the other hand we have these equalities.

$$\begin{aligned}\int_{-1}^1 R(x)\,dx &= \sum_{i=0}^{n} R(x_i) \\ &= \sum_{i=0}^{n} P_{n+1}(x_i) \cdot Q(x_i) + R(x_i)\end{aligned}$$

The first equality is because the degree of $R(x)$ is less than the number of points used in the quadrature formula. The second equality is because the points x_i were chosen to be the zeros of P_{n+1}. So, we have shown that the quadrature formula is exact for polynomials of degree $\leq 2n + 1$. □

3 Implementation of Gaussian Quadrature

For a given Legendre polynomial $P_{n+1}(x)$, finding the roots is not such an easy task. So, the way this is usually done is to determine these values $\{x_0, x_1, \ldots, x_n\}$ for a certain value of n and determine the coefficients $\{a_0, a_1, \ldots, a_n\}$. Then store these values and call them from memory or embed the values in a program implementing Gaussian Quadrature. Here are the x_i and a_i values for $n = 9$.

$$\begin{aligned}
x_4 &= -0.14887434\\
x_5 &= 0.14887434\\
x_3 &= -0.43339539\\
x_6 &= 0.43339539\\
x_2 &= -0.67940957\\
x_7 &= 0.67940957\\
x_1 &= -0.86506337\\
x_8 &= 0.86506337\\
x_0 &= -0.97390653\\
x_9 &= 0.97390653
\end{aligned}$$

The coefficients are the following.

$$\begin{aligned}
a_4 = a_5 &= .29552422\\
a_3 = a_6 &= .26926672\\
a_2 = a_7 &= .21908636\\
a_1 = a_8 &= .14945135\\
a_0 = a_9 &= .06667134
\end{aligned}$$

If we used the exact values of these points and weights, then the Gaussian Quadrature formula would be exact for polynomials of degree ≤ 19. In the next section we implement a program with fewer points just for convenience.

4 TI-89 Program for Gaussian Quadrature

Here is a program with eight points, $n = 7$. The points and their weights are given below. Be careful to enter the values correctly. Obviously, the program will not give good estimates if the numbers are not correct.

$$
\begin{array}{ll}
x_0 = -.9602898564975363 & a_0 = .1012285362903706 \\
x_1 = -.7966664774136267 & a_1 = .2223810344533744 \\
x_2 = -.5255324099163290 & a_2 = .3137066458778874 \\
x_3 = -.1834346424956498 & a_3 = .3626837833783621 \\
x_4 = .1834346424956498 & a_4 = .3626837833783621 \\
x_5 = .5255324099163290 & a_5 = .3137066458778874 \\
x_6 = .7966664774136267 & a_6 = .2223810344533744 \\
x_7 = .9602898564975363 & a_7 = .1012285362903706
\end{array}
$$

```
:gaussq(f,a,b)
:Prgm
:[-.9602898564975363, -.7966664774136267, -.5255324099163290, -.1834346424956498,
.1834346424956498, .5255324099163290, .7966664774136267, .9602898564975363] → points
:[.1012285362903706, .2223810344533744, 3137066458778874, .3626837833783621, .3626837833783621,
.3137066458778874, .2223810344533744, .1012285362903706] → weights
:0 → p
:For i,1,8
:p + (b-a)/2*weights[1,i]*f | (x=(b-a)/2*points[1,i] + (b+a)/2) → p
:EndFor
:EndPrgm
```

The variable f is the function with assumed variable x. The inputs a and b are the left and right endpoints of the interval of integration, respectively. The value of the integral is stored in the variable p. This program will give the exact integral for polynomials up to degree fifteen except for round-off error. So, to test the program compute the following integral.

$$\int_0^1 x^{15}\,dx = \frac{1}{16} = .0625$$