HOMOGENEOUS DIFFERENTIAL EQUATIONS

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In this post we give the basic theory of homogeneous differential equations. These equations can be put in the following form.

$$\frac{dy}{dx} = G\left(\frac{y}{x}\right) \tag{1}$$

The function $G(z)$ is such that substituting $\frac{y}{x}$ for z gives the right hand side of (1).

There is a test to verify that a differential equation is homogeneous. Substitute $t \cdot x$ for x and $t \cdot y$ for y in the differential equation. If t can be eliminated from the equation, then the equation is homogeneous.

1. Solving a Homogeneous Equation

Consider an equation that has the form of (1). Substitute $v = \frac{y}{x}$ so that the right hand side of (1) becomes $G(v)$. By letting $v = \frac{y}{x}$ we get that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. This gives us the following differential equation in v and x.

$$v + x\frac{dv}{dx} = G(v) \tag{2}$$

Now (2) is separable and has the form

$$\frac{dv}{G(v) - v} = \frac{dx}{x} \tag{3}$$

So, the solution is obtained by solving (3) to get $v = h(x, C)$. Then substitute $\frac{y}{x}$ for v to get

$$\frac{y}{x} = h(x, C). \tag{4}$$

2. An Example

Consider the following differential equation.

$$(y^2 + x^2)\, dx - x^2\, dy = 0 \tag{5}$$

This can be put in the following form.

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 1 \tag{6}$$

From (2) we can write this as

$$v + x\frac{dv}{dx} = v^2 + 1. \tag{7}$$

Finally, we get

$$\frac{dv}{v^2 - v + 1} = \frac{dx}{x}. \tag{8}$$

Our solution is thus

$$\ln(x) = \frac{2\tan^{-1}\left(\frac{2v-1}{\sqrt{3}}\right)}{\sqrt{3}} + C. \tag{9}$$

We now substitute $\frac{y}{x}$ back for v to get

$$\ln(x) = \frac{2\tan^{-1}\left(\frac{2\left(\frac{y}{x}\right)-1}{\sqrt{3}}\right)}{\sqrt{3}} + C. \tag{10}$$

There are several other methods that are similar to this where a substitution of a variable for some expression of x and y is made that makes the differential equation separable.