HOMOGENEOUS DIFFERENTIAL EQUATIONS

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In this post we give the basic theory of homogeneous differential equations. These equations can be put in the following form.

\begin{equation}
\frac{dy}{dx} = G \left( \frac{y}{x} \right)
\end{equation}

The function \( G(z) \) is such that substituting \( \frac{y}{x} \) for \( z \) gives the right hand side of (1).

There is a test to verify that a differential equation is homogeneous. Substitute \( t \cdot x \) for \( x \) and \( t \cdot y \) for \( y \) in the differential equation. If \( t \) can be eliminated from the equation, then the equation is homogeneous.

1. Solving a Homogeneous Equation

Consider an equation that has the form of (1). Substitute \( v = \frac{y}{x} \) so that the right hand side of (1) becomes \( G(v) \). By letting \( v = \frac{y}{x} \) we get that \( \frac{dy}{dx} = v + x \frac{dv}{dx} \). This gives us the following differential equation in \( v \) and \( x \).

\begin{equation}
v + x \frac{dv}{dx} = G(v)
\end{equation}

Now (2) is separable and has the form

\begin{equation}
\frac{dv}{G(v) - v} = \frac{dx}{x}
\end{equation}

So, the solution is obtained by solving (3) to get \( v = h(x, C) \). Then substitute \( \frac{y}{x} \) for \( v \) to get

\begin{equation}
\frac{y}{x} = h(x, C).
\end{equation}

2. An Example

Consider the following differential equation.

\begin{equation}
(y^2 + x^2) \, dx - x^2 \, dy = 0
\end{equation}

This can be put in the following form.
(6) \[ \frac{dy}{dx} = \left( \frac{y}{x} \right)^2 + 1 \]

From (2) we can write this as

(7) \[ v + x \frac{dv}{dx} = v^2 + 1. \]

Finally, we get

(8) \[ \frac{dv}{v^2 - v + 1} = \frac{dx}{x}. \]

Our solution is thus

(9) \[ \ln(x) = \frac{2 \tan^{-1} \left( \frac{2v-1}{\sqrt{3}} \right)}{\sqrt{3}} + C. \]

We now substitute \( \frac{y}{x} \) back for \( v \) to get

(10) \[ \ln(x) = \frac{2 \tan^{-1} \left( \frac{2 \left( \frac{y}{x} \right) - 1}{\sqrt{3}} \right)}{\sqrt{3}} + C. \]

There are several other methods that are similar to this where a substitution of a variable for some expression of \( x \) and \( y \) is made that makes the differential equation separable.