LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS

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In this post we give solution of the most general first-order ordinary differential equation. This equation has the form:

$$\frac{dx}{dt} + p(t)\,x = g(t). \tag{1}$$

We solve the equation by finding an integrating factor in much the same way that we did to produce exact differential equations from equations that were not exact.

Consider the following expression.

$$\frac{d}{dt}\left(\exp\left(\int p(t)dt\right)\cdot x\right) = \exp\left(\int p(t)dt\right)\cdot\left(\frac{dx}{dt} + p(t)\cdot x\right) \tag{2}$$

We can put (1) and (2) together in the following way.

$$\frac{d}{dt}\left(\exp\left(\int p(t)dt\right)\cdot x\right) = \exp\left(\int p(t)dt\right)\cdot\left(\frac{dx}{dt} + p(t)\,x\right) = \exp\left(\int p(t)dt\right)\cdot g(t)$$

We can now solve.

$$\exp\left(\int p(t)dt\right)\cdot x = \int\left(\exp\left(\int p(t)dt\right)\cdot g(t)\right)dt + C$$

$$x = \exp\left(-\int p(t)dt\right)\cdot\left(\int\left(\exp\left(\int p(t)dt\right)\cdot g(t)\right)dt + C\right)$$

The basic trick is to note that the function $\exp\left(\int p(t)dt\right)$ is an integrating factor for the left-hand side of (1).

1. Example

Consider the differential equation.

$$\frac{dx}{dt} + \frac{1}{t}\,x = \sin(t) \tag{3}$$

Our integrating factor is $\exp\left(\int \frac{1}{t}\,dt\right) = \exp(\ln(t)) = t$. So, our differential equation becomes

$$\frac{d}{dt}(t \cdot x) = t \cdot \sin(t)$$

and the solutiion is

$$x(t) = \frac{1}{t} \int t \, \sin(t) \, dt + \frac{C}{t} = \frac{\sin(t)}{t} - \cos(t) + \frac{C}{t}.$$