LINEAR INDEPENDENCE, THE WRONSKIAN, AND VARIATION OF PARAMETERS

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In this post we determine when a set of solutions of a linear differential equation are linearly independent. We first discuss the linear space of solutions for a homogeneous differential equation.

1. Homogeneous Linear Differential Equations

We start with homogeneous linear n^{th}-order ordinary differential equations with general coefficients. The form for the n^{th}-order type of equation is the following.

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_0(t)x = 0 \tag{1}$$

It is straightforward to solve such an equation if the functions $a_i(t)$ are all constants. However, for general functions as above, it may not be so easy. However, we do have a principle that is useful. Because the equation is linear and homogeneous, if we have a set of solutions $\{x_1(t), \ldots, x_n(t)\}$, then any linear combination of the solutions is also a solution. That is

$$x(t) = C_1x_1(t) + C_2x_2(t) + \cdots + C_nx_n(t) \tag{2}$$

is also a solution for any choice of constants $\{C_1, C_2, \ldots, C_n\}$. Now if the solutions $\{x_1(t), \ldots, x_n(t)\}$ are *linearly independent*, then (2) is the general solution of the differential equation. We will explain why later.

What does it mean for the functions, $\{x_1(t), \ldots, x_n(t)\}$, to be *linearly independent*? The simple straightforward answer is that

$$C_1x_1(t) + C_2x_2(t) + \cdots + C_nx_n(t) = 0 \tag{3}$$

implies that $C_1 = 0$, $C_2 = 0$, $\ldots$, and $C_n = 0$ where the C_i's are arbitrary constants.

This is the definition, but it is not so easy to determine from it just when the condition holds to show that a given set of functions, $\{x_1(t), x_2(t), \ldots, x_n\}$, is linearly independent. The *Wronskian* is a practical way of determining this.

Let $\{x_1(t), x_2(t), \ldots, x_n\}$ be an arbitrary set of functions that are $(n-1)$ times continuously differentiable. Then the Wronskian matrix is given by the following.

$$W(x_1(t), x_2(t), \ldots, x_n(t)) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \ddots & & \vdots \\ x^{(n-1)}(t) & x_2'(t) & \cdots & x_n^{(n-1)}(t) \end{bmatrix} \tag{4}$$

It turns out that the original set of functions, $\{x_1(t), x_2(t), \ldots, x_n\}$, is linearly independent if and only if $\det W(x_1(t), x_2(t), \ldots, x_n) \neq 0$. This would still be a difficult task, but computer technology can come to our aid. In your set of programs is a program that produces the Wronskian matrix. The calculations are symbolic and the determinant program in the TI-Nspire CX CAS will also do that calculation symbolically. This gives us a quick and reliable means of determining when a set of functions is linearly independent.

2. Example

Suppose that our set of functions is given by $\{\sin(t), \cos(t), \exp(t)\}$. Using our program we get that the Wronskian matrix is given by

$$W(\sin(t), \cos(t), \exp(t) = \begin{bmatrix} \sin(t) & \cos(t) & \exp(t) \\ \cos(t) & -\sin(t) & \exp(t) \\ -\sin(t) & -\cos(t) & \exp(t) \end{bmatrix} \tag{5}$$

By computing the determinant of this matrix we get

$$\det\left(W(\sin(t), \cos(t), \exp(t))\right) = -2\exp(t) \tag{6}$$

which indicates that these functions are linearly independent.

3. Proof

We will now show that if the Wronskian of a set of functions is not zero, then the functions are linearly independent. As above suppose that $\{x_1(t), x_2(t), \ldots, x_n\}$ is our set of functions which are $(n-1)$ times continuously differentiable. Consider a linear combination of these functions as given in (2). We would like to determine if the constants $\{C_1, \ldots, C_n\}$ must all be zero or not. Consider the equation $C_1x_1(t) + \cdots + C_nx_n(t) = 0$. If this holds then by taking successive derivatives, all of the equations below also hold.

$$\begin{gathered} C_1 \cdot x_1(t) + C_2 \cdot x_2(t) + \cdots + C_n \cdot x_n(t) = 0 \\ C_1 \cdot x_1'(t) + C_2 \cdot x_2'(t) + \cdots + C_n \cdot x_n'(t) = 0 \\ \vdots \\ C_1 \cdot x_1^{(n-1)}(t) + C_2 \cdot x_2^{(n-1)}(t) + \cdots + C_n \cdot x_n^{(n-1)}(t) = 0 \end{gathered} \tag{7}$$

We can rewrite the equations in matrix form as the following.

$$\begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \ddots & & \vdots \\ x^{(n-1)}(t) & x_2'(t) & \cdots & x_n^{(n-1)}(t) \end{bmatrix} \times \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{8}$$

Note that our matrix of coefficients for the equation is just the Wronskian of our set of functions $\{x_1(t), x_2(t), \dots, x_n\}$. If the Wronskian is not zero, then there is a unique solution to the equations, namely, $C_i = 0$ for all $i = 1, 2, \dots, n$. On the other hand, if the Wronskian is zero, then there are infinitely many solutions.

Note also that we only need that the Wronskian is not zero for some value of $t = t_0$. Since all the functions in the Wronskian matrix are continuous, the Wronskian will be non-zero in an interval about t_0 as well. Suppose that our functions are all solutions of an n^{th} degree linear differential equation. Suppose also that we want a solution $x(t)$ such that $x^{(i)}(t_0) = A_i$ for some set of values $\{A_0, A_1, \dots, A_{(n-1)}\}$. Suppose that the determinant of the Wronskian matrix is non-zero at t_0. Then there will be a solution $x(t) = C_1x_1(t) + C_2x_2(t) + \cdots + C_nx_n(t)$ such that $x^{(i)}(t_0) = A_i$ for all $i = 0, 1, \dots, n-1$. The solution is given by the set of constants that satisfy the following equation at t_0.

$$\begin{bmatrix} x_1(t_0) & x_2(t_0) & \cdots & x_n(t_0) \\ x_1'(t_0) & x_2'(t_0) & \cdots & x_n'(t_0) \\ \vdots & \ddots & & \vdots \\ x^{(n-1)}(t_0) & x_2'(t_0) & \cdots & x_n^{(n-1)}(t_0) \end{bmatrix} \times \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{n-1} \end{bmatrix} \tag{9}$$

For this set of constants we have a solution $x(t) = C_1x_1(t) + C_2x_2(t) + \cdots + C_nx_n(t)$ valid in a neighborhood of t_0 such that $x^{(i)}(t_0) = A_i$ for all $i = 0, 1, \dots, n-1$.

4. Example

Consider the differential equation

$$x'' + x = 0 \tag{10}$$

Suppose that we have two initial conditions $x(0) = 1$ and $x'(0) = -1$ that we want satisfied.

We have two solutions to the general equation, $\sin(t)$ and $\cos(t)$. So, we also have any linear combination, $x(t) = C_1 \sin(t) + C_2 \cos(t)$ of these solutions as a solution as well. We form the Wronskian matrix from our solutions.

$$W(\sin(t), \cos(t)) = \begin{bmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix} \tag{11}$$

Evaluating this at $t = t_0 = 0$ we get the matrix

$$W(\sin(t), \cos(t)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{12}$$

which has non-zero determinant. The initial conditions give us $A_0 = 1$ and $A_1 = -1$. So, we apply (9) to solve for C_1 and C_2 to get that $C_1 = -1$ and $C_2 = 1$ to get the solution fo the initial value problem of (10).

$$x(t) = \cos(t) - \sin(t) \tag{13}$$

5. The General Solution of the Homogeneous Linear Differential Equation of Order n

We have hinted that the general solution of (1) is a linear combination of linearly independent solutions of (1). Suppose that we have solutions $\{x_1(t), \ldots, x_n(t)\}$ such that the determinant of the Wronskian matrix for these solutions is not zero at a point t_0. Then there are constants $\{C_1, \ldots, C_n\}$ so that the initial conditions $x(t_0) = A_0, x'(t_0) = A_1, \ldots, x^{(n-1)}(t_0) = A_{(n-1)}$ are satisfied using (9). This is because we are assuming that the determinant of the Wronskian matrix at t_0 is not zero. On the other hand, if we have a solution of an $n^{\text{th}}-$order equation, call if $x(t)$ and we know the values $x(t_0), x'(t_0), x''(t_0, \ldots, x^{(n-1)}(t_0)$, then our argument above says that this solution will be unique. However, there is a solution of the form $x(t) = C_1x_1(t) + C_2x_2(t) + \cdots + C_nx_n(t)$. So, every solution is of this form.

So, find n solutions of (1), $\{x_1(t), \ldots, x_n(t)\}$. Determine that they are linearly independent using the Wronskian. Then the space of all linear combinations of these solutions, $C_1x_1(t) + \cdots + C_nx_n(t)$, is the collection of all solutions.

6. Variation of Parameters

In this section we give another use of the Wronskian matrix. We start with the general $n^{\text{th}}-$order linear differential equation. It has the following form.

$$\frac{d^nx}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_0(t)x = g(t) \tag{14}$$

Note that we are assuming that the leading coefficient function $a_n(t) \equiv 1$. There is no loss of generality in doing this, but it makes the calculations easier. Suppose that we have a set of linearly independent solutions, $\{x_1(t), x_2(t), \ldots, x_n(t)\}$, of the related homogeneous equation.

$$\frac{d^nx}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_0(t)x = 0 \tag{15}$$

Then we know that the general solution to (14) will be of the form

$$x_0(t) + C_1x_1(t) + \cdots + C_nx_n(t) \tag{16}$$

where $x_0(t)$ is a particular solution to (14) and $C_1x_1(t) + \cdots + C_nx_n(t)$ is the general solution to (15).

Now we assume that there is a particular solution of the form $x_0 = v_1(t)x_1(t) + \cdots + v_n(t)x_n(t)$. We now describe a method of determining a set of functions, $\{v_1(t), v_2(t), \ldots, v_n(t)\}$, that will give such a solution. When we take the derivative of this function we get

$$\begin{aligned} \frac{dx_0}{dt} &= \frac{d}{dt}(v_1x_1 + \cdots + v_nx_n) \\ &= v_1'x_1 + \cdots + v_n'x_n + v_1x_1' + \cdots + v_nx_n' \end{aligned} \tag{17}$$

and we arbitrarily set $v_1'x_1 + \cdots + v_n'x_n = 0$ to leave us with $\frac{dx_0}{dt} = \frac{d}{dt}(v_1x_1 + \cdots + v_nx_n) = v_1x_1' + \cdots + v_nx_n'$. When we take the second derivative of our particular solution we get

$$\begin{aligned} \frac{d^2x_0}{dt^2} &= \frac{d^2}{dt^2}(v_1x_1 + \cdots + v_nx_n) \\ &= \frac{d}{dt}(v_1x_1' + \cdots + v_nx_n') \\ &= v_1'x_1' + \cdots + v_n'x_n' + v_1'x_1'' + \cdots + v_n'x_n'' \end{aligned} \tag{18}$$

and again we arbitrarily set $v_1'x_1' + \cdots + v_n'x_n' = 0$. We continue in this fashion until we end up with the following set of equations.

$$\begin{aligned} 0 &= v_1'x_1 + \cdots + v_n'x_n \\ 0 &= v_1'x_1' + \cdots + v_n'x_n' \\ &\vdots \\ g(x) &= v_1'x_1^{(n-1)} + \cdots + v_n'x_n^{(n-1)} \end{aligned} \tag{19}$$

We can rewrite this in matrix form as

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_1' & x_2' & \cdots & x_n' \\ \vdots & \ddots & & \vdots \\ x^{(n-1)} & x_2^{(n-1)} & \ldots & x_n^{(n-1)} \end{bmatrix} \times \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{bmatrix} \tag{20}$$

The solution to this matrix equation gives us a vector whose entries are the derivatives of our unknown functions $v_i(t)$. The matrix on the left is just the Wronskian. By assumption, its determinant is not zero for any t in the interval under consideration. So, the equations can be solved for a unique set of functions $\{v_1', \ldots, v_n'\}$. After integrating these we get our particular solution $x_0 = v_1x_1 + v_2x_2 + \cdots + v_nx_n$.

7. Example

Consider the differential equation

$$x''' - x'' + x' - x = \exp(2t) \tag{21}$$

We find the general solution of the homogeneous equation and find it to be $C_1 \sin(t) + C_2 \cos(t) + C_3 \exp(t)$. For the functions $\{\sin(t), \cos(t), \exp(t)\}$ we get the Wronskian matrix to be the following.

$$\begin{bmatrix} \sin(t) & \cos(t) & \exp(t) \\ \cos(t) & -\sin(t) & \exp(t) \\ -\sin(t) & -\cos(t) & \exp(t) \end{bmatrix} \tag{22}$$

We now use (20) to solve for the $\{v_1', v_2', v_3'\}$. Integrating these and combining them we get the solution $x_0 = v_1x_1 + v_2x_2 + v_3x_3 = \frac{\exp(2t)}{2}$. It would be a good idea to do this using the steps described. This help assure that you understand the process involved. However, there is a program provided for you for the TI-Nspire CX CAS which will do all of this. Here is the call for the program.

$$\text{VarParameters}([\sin(t), \cos(t), \exp(t)], \exp(2t))$$

With this particular solution we now have the following general solution.

$$x(t) = C_1 \sin(t) + C_2 \cos(t) + C_3 \exp(t) + \frac{\exp(2t)}{2} \tag{23}$$

For this example, we could also have used the method of *undetermined coefficients* as well to get the same outcome. The method of variation of parameters works for any function $g(t)$. The method of undetermined coefficients only works for a limited set of functions $g(t)$.