Solution of Linear Systems of Ordinary Differential Equations

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1 Linear Ordinary Differential Equations

Consider a first-order linear system of differential equations with constant coefficients. This can be put into matrix form.

$$\begin{aligned} \frac{dx}{dt} &= Ax \\ x(0) &= C \end{aligned} \tag{1}$$

Here $x(t)$ is a vector function expressed as a column vector, $x : R \to R^n$ and A is an $n \times n$ matrix. The solution of this can be obtained by using what is called the *exponential of a matrix*. For the $n \times n$ matrix A, define

$$\exp(t \cdot A) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

This matrix series will converge for all values of t. For each value of t, the limit $\exp(tA)$ is an $n \times n$ matrix. Furthermore,

$$\exp((t+s)A) = \exp(tA) \cdot \exp(sA)$$

and

$$\frac{d \exp(tA)}{dt} = A \cdot \exp(tA).$$

The above features are similar to the scalar value exponential function. On the other hand, the matrix exponential function does have some surprising differences. Among these is that it may be the case that for two $n \times n$ matrices A and B, it may be the case that $\exp(A + B) \neq \exp(A) \cdot \exp(B)$.

2 Matrix Solution of the Equation

The Picard method shows that a linear system of differential equations has a unique solution. However, we can easily produce the solution to (1) using the matrix exponential function. The solution is obvious from the above formulas.

$$x(t) = \exp(tA) \cdot C$$

This is not just a cute theoretical formula. It is a practical way of solving linear differential equations because it brings all of the computational techniques of linear algebra to bear. Here is a simple case. Suppose that P is an invertible $n \times n$ matrix and that $PAP^{-1} = B$. One can easily see that $\exp(tB) = P\exp(tA)P^{-1}$. Suppose, for instance, that A is diagonalizable so that $A = PDP^{-1}$ where D is a diagonal matrix. Then, $\exp(tA) = P\exp(tD)P^{-1}$. However, one can easily compute the exponential of a diagonal matrix. If

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & d_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & d_n \end{bmatrix},$$

then

$$\exp(tD) = \begin{bmatrix} \exp(td_1) & 0 & 0 & \cdots & 0 \\ 0 & \exp(td_2) & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & \exp(td_{n-1}) & 0 \\ 0 & 0 & \cdots & 0 & \exp(td_n) \end{bmatrix}.$$

Thus, for diagonalizable matrices A we have a very practical means of computing the solution of (1). In fact, matrix methods can be applied to solve (1) for all cases using the Jordan Canonical Form of a general $n \times n$ matrix A. The eigenvalues of A play an important role in understanding the behavior of the solutions of (1).

3 Approximation of the Exponential of a Matrix

The exponential of a matrix can be determined exactly for many cases and has powerful implementation in many programs such as Maple and Mathematica. On the other hand, we can implement a finite approximation of the exponential of a matrix by brute force. Let A be an $m \times m$ matrix. Then $\exp(tA)$ can be approximated by stopping after K terms in the exponential series. The calculation will be carried out whether the entries in A are floating point numbers, exact values, or symbols. There is no need for a program. The right hand side below can be entered in the command line of the home screen of the TI-89.

$$\exp(tA) \approx \sum_{n=0}^{K} \frac{1}{n!}(t \cdot A)^n$$

If we use this approximation in the formula for the solution of (1),

$$x(t) = \exp(tA) \cdot C,$$

then we are getting a truncation of the power series solution of $x(t)$.

4 Eigenvalues

The eigenvalues of the matrix A are important in determining the exact exponential of the matrix. They are also important in determining the qualitative solution of the differential equation (1).

The eigenvalues are solutions of the following equation.

$$A \cdot v = \lambda \cdot v$$

In the above, λ is a real number, an *eigenvalue* for the matrix A. The symbol v is a non-zero vector, an *eigenvector for* λ. From the definition of λ and v, we get that

$$(A - \lambda \cdot I) \cdot v = 0$$

as a vector equation where I is the $m \times m$ identity matrix. This implies that

$$\det(A - \lambda \cdot I) = 0.$$

If we treat λ as a symbol, then $\det(A - \lambda \cdot I)$ is a polynomial in λ, the *characteristic polynomial* of the matrix A. The roots of this polynomial are known as the *generalized eigenvalues* for A.

In the special case that the generalized eigenvalues are all real and distinct, then the matrix A will be diagonizable and the solution to the differential equation (1) can be solved easily using matrix methods. We state the formal result next.

Theorem 4.1. *Suppose that all the eigenvalues of A are real and distinct. Then there is a non-singular matrix P and a diagonal matrix D such that $A = PDP^{-1}$. The diagonal entries of D are the eigenvalues of A.*

It should also be said that the matrix P^{-1} in Theorem 4.1 has as its columns the eigenvectors for the eigenvalues of A. The eigenvector columns need to be in the same order as the eigenvalues in the diagonal of the matrix D.

Example 4.2. *Solve the following linear ordinary differential equations using matrix methods.*

$$\frac{d^2x}{dt^2} + x = 0 \tag{2}$$

and

$$\frac{d^2x}{dt^2} - x = 0 \tag{3}$$

In case (2) above, we make the substitution $y = \frac{dx}{dt}$ to get the following first order system.

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x$$

The matrix equation then becomes the following.

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

By the definition of the exponential of a matrix and using power series identities for $\cos t$ and $\sin t$ we get

$$\exp\left(t \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

Thus our solution in the first case is $x(t) = C_1 \cos t + C_2 \sin t$.

In case (3) above, we go through the same steps and get the following matrix equation.

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

In this case we use the fact that the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is diagonalizable with eigenvalues $\{1, -1\}$. We get the following conjugacy.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Thus,

$$\exp\left(t \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

So, in this case our solution is $x(t) = \frac{C_1 + C_2}{2} \cdot e^t + \frac{C_1 - C_2}{2} \cdot e^{-t}$.