LINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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In this post we determine solution of the linear 2^{nd}-order ordinary differential equations with constant coefficients.

1. The Homogeneous Case

We start with homogeneous linear 2^{nd}-order ordinary differential equations with constant coefficients. The form for the 2^{nd}-order equation is the following.

$$a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x = 0 \tag{1}$$

The solution is determined by supposing that there is a solution of the form $x(t) = e^{mt}$ for some value of m. When we substitute a solution of this form into (1) we get the following equation.

$$a_2m^2e^{mt} + a_1me^{mt} + a_0e^{mt} = 0 \tag{2}$$

By cancelling e^{mt} we get the following algebraic equation.

$$a_2m^2 + a_1m + a_0 = 0 \tag{3}$$

We now consider the possible types of solutions for (3) we might have and see what solutions we get for (1) for each of these types. The first type of solution that we may get is a real root of order one, m_1. In this case we get a soluton, e^{m_1t} to the differential equation. If we had two distinct such roots, $m_1 \neq m_2$, then $C_1e^{m_1} + C_2e^{m_2t}$ would also be a solution for any constants C_1 and C_2.

$$x(t) = C_1e^{m_1t} + C_2e^{m_2t} \tag{4}$$

where C_1 and C_2 are arbitrary constants to be determined by the initial conditions of the problem.

If a solution of (3) is a single repeated root of order 2, m_1, then the solution is of the form

$$x(t) = C_1e^{m_1t} + C_2te^{m_1t} \tag{5}$$

where C_1 and C_2 are arbitrary constants to be determined by the initial conditions of the problem.

If the solution of (3) is a complex number, $a + bi$, then the complex conjugate, $a - bi$ is also a solution. The solution of the differential equation is of the form

$$x(t) = e^{at}(C_1 \sin(bt) + C_2 \cos(bt)) \tag{6}$$

where C_1 and C_2 are arbitrary constants to be determined by the initial conditions of the problem.

So, we see that there are distinct types of solutions that we my have to the differential equation (1). These are given by combinations of solutions of the type (4), (5), or (6) depending on the types of roots we have to (3). The general solution of the n^{th}-order homogeneous case will be a linear combination of n linearly independent solutions.

2. The General Case

Now we consider is when the equation is not homogeneous. The form of the equation then becomes the following.

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = g(x) \tag{7}$$

Suppose that we have a specific solution of this equation. Let us call it $x_0(t)$. Now suppose that we also have the general solution of the related homogeneous equation (1). This will have the form $C_1 x_1(t) + C_2 x_2(t)$. It is clear that the sum of these two solutions is a solution of (7). Thus, (8) will be the general solution of (7).

$$x(t) = x_0 + C_1 x_1(t) + C_2 x_2(t) \tag{8}$$

3. Example

Consider the following differential equation.

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 5 \tag{9}$$

The related homogeneous equation is below.

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 0 \tag{10}$$

Clearly a specific solution to (9) is $x_0(t) = 5$. Solving for the general solution of (10) we get $C_1 e^t + C_2 t e^t$. So, the general solution of (9) is $x(t) = 5 + C_1 e^t + C_2 t e^t$.