Logarithm and Exponential Functions

We want to give a precise definition for the logarithm and derive its properties. The exponential function is the inverse function for the logarithm. Based on properties of the logarithm, the properties of the exponential function then follow.

Logarithm.

We define the logarithm by the following formula.

(1) $\ln(x) = \int_1^x \frac{1}{t}\,dt \quad \text{for all } x > 0$

The following two properties follow immediately.

(2) $\frac{d}{dx}\ln(x) = \frac{1}{x}$

(3) $\ln(1) = 0$

We now show the following property.

(4) $\ln(x \cdot y) = \ln(x) + \ln(y)$

Let $a > 0$ be any constant. Consider the function $\ln(a \cdot x)$. If we compute the derivative we get $\frac{d}{dx}\ln(a \cdot x) = \frac{1}{x}$. Since this is also the derivative of $\ln(x)$, these two functions must differ by a constant. So, $\ln(a \cdot x) \equiv \ln(x) + C$. We can determine the value of C by letting $x = 1$. With that substitution we get $\ln(a) = \ln(1) + C = 0 + C = C$. Consequently, $\ln(a \cdot x) = \ln(a) + \ln(x)$. This proves formula (4).

(5) $\ln(x)$ is an increasing function

This follows from the fact that $\frac{d}{dx}\ln(x) = \frac{1}{x} > 0$ for all $x > 0$.

(6) $\lim_{x \to \infty} \ln(x) = \infty$.

To show this, we only need to give an example of a sequence of points $x_n \to \infty$ such that $\ln(x_n) \to \infty$. Let $x_n = 2^n$. Then $\lim_{n \to \infty} 2^n = \infty$. However, $\ln(2^n) = n \ln 2$ by property (4) above. But, $\ln(2) > 0$ by definition (1). So, $\lim_{n \to \infty} n \ln 2 = \infty$. This proves (6).

(7) $\lim_{x \to 0^+} \ln(x) = -\infty$

To show this, we only need to give an example of a sequence of points $x_n \to 0^+$ such that $\ln(x_n) = -\infty$. Let $x_n = 2^{-n}$. Then $x_n \to 0$. But $\ln(2^{-n}) = n \ln(2^{-1}) = n \cdot (-\ln 2)$. So, $\lim_{n \to \infty} \ln(2^{-n}) = -\infty$ as required.

Exponential.

We define the exponential function, $\exp(x)$, to be the inverse function of the logarithm. The following properties then follow from this definition.

(1) $\exp(x)$ is defined on $(-\infty, \infty)$

(2) $\exp(x) > 0$ for all x

(3) $\exp(x)$ is an increasing function for all x

(4) $\lim_{x \to \infty} \exp(x) = \infty$

(5) $\lim_{x \to -\infty} \exp(x) = 0$

(6) $\frac{d}{dx} \exp(x) = \exp(x)$

This last property follows from the derivative of the inverse function of a function. $\frac{d}{dx} \exp(x) = \frac{1}{\frac{d}{dy} \ln y}$ where $\ln y = x$. This gives $\frac{d}{dx} \exp(x) = \frac{1}{1/y} = y = \exp(x)$.

(7) $\exp(x + y) = \exp(x) \cdot \exp(y)$

This follows from (4) for the logarithm.

Power Function a^x

The function a^x is defined for all x and for all $a > 0$. The simplest way to understand this function is to define it by $a^x = \exp(x \ln a)$. We can then immediately see the calculus properties of this function.

(1) $\frac{d}{dx} a^x = \frac{d}{dx} \exp(x \cdot \ln a) = \exp(x \cdot \ln a) \cdot \ln a = (\ln a) \cdot a^x$

(2) $\frac{d}{dx}a^x = a^x$ if and only if $\ln a = 1$. This leads to the definition of the number e as that number having the property that $\ln e = 1$.

(3) $a^{x+y} = a^x \cdot a^y$

We can now calculate derivatives and limits of complicated functions involving powers. For instance:

$$\frac{d}{dx}f(x)^{g(x)} =$$

$$\exp(g(x)\cdot\ln(f(x))) = \exp(g(x)\cdot\ln(f(x)))\cdot\left(g'(x)\cdot\ln(f(x)) + \frac{g(x)\cdot f'(x)}{f(x)}\right)$$

Of course, for the formula to be valid, we must have $f(x) > 0$.

Examples.

1. $\frac{d}{dx}x^x = \frac{d}{dx}\exp(x\cdot\ln x) = \exp(x\cdot\ln x)\cdot(\ln x + 1) = (1+\ln x)\cdot x^x$

2. $\frac{d}{dx}x^a = \frac{d}{dx}\exp(a\cdot\ln x) = \exp(a\cdot\ln x)\cdot\left(0\cdot\ln x + \frac{a}{x}\right) = \frac{a}{x}\cdot x^a = a\cdot x^{a-1}$

3. Atmospheric pressure is given by the formula $p_0 \cdot a^h$ where p_0 is the pressure at sea level and h is the altitude above sea level. Some consistent set of units is assumed. Suppose that the pressure at 18,000 feet is one-half that at sea level. Express the formula using this information.

$$p_0 \cdot a^{18000\,ft} = \frac{1}{2}\cdot p_0$$

$$a^{18000\,ft} = \frac{1}{2}$$

$$18000\,ft\cdot\ln a = -\ln 2$$

$$\ln a = \frac{-\ln 2}{18000\,ft}$$

$$a = \exp\left(\frac{-\ln 2}{18000\, ft}\right) = e^{\frac{-\ln 2}{18000\, ft}}$$

$$p_0 \cdot a^h = p_0 \cdot \left(\frac{1}{2}\right)^{h/18000\, ft}$$

4. A filter of ten inches in length will filter out $\frac{3}{4}$ of a given contaminant from water. How much will a filter of length fifteen inches filter out?

From the derivation above, we have the following formula for general length *L*.

$$c_L = c_0 \cdot \left(\frac{3}{4}\right)^{L/10\, in}$$

For the problem at hand, $L = 15\ inches$.

$$c_{15} = c_0 \cdot \left(\frac{3}{4}\right)^{15/10} = c_0 \cdot \frac{3\sqrt{3}}{8}$$