MAA 4211 FINAL - JAMES KEESLING

NAME _____

Work all problems and show all work. Each problem is worth 10 points. Partial credit will be given for correct reasoning even if the final answer is incorrect. Credit will be deducted for incorrect reasoning even if the final answer is correct.

Problem 1. Suppose that X and Y are metric spaces and that $A \subset X$ is compact. Suppose that $f : X \to Y$ is continuous. Show that $f(A)$ is compact.

Problem 2. Show that $[0, 1]$ is uncountable.

Problem 3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and suppose that $f(z) = 0$. Suppose also that $f'(z) \neq 0$. Define $g(x) = x - \frac{f(x)}{f'(x)}$. Show that z is an attracting fixed point for $g(x)$.

Problem 4. Show that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Problem 5. Consider $f(x) = x^2 \cdot \sin\left(\frac{1}{x}\right)$ for all $x \neq 0$. Define $f(0) = 0$. Show that $f(x)$ is differentiable for all $x \in \mathbb{R}$ and that $f'(x)$ is not continuous at 0.

Problem 6. State and prove the Banach Fixed Point Theorem.

Problem 7. State and prove **Cauchy's Mean Value Theorem**.

Problem 8. Show that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for all $|x| < 1$.

Problem 9. Suppose that X is a complete metric space and that $A \subset X$ is closed. Show that A is a complete metric space.

Problem 10. Determine numerical solutions of the following equations.

$$\cos(x) = x$$

$$x^{15} + 3x^4 - 4x^3 + x = 5$$