ADVANCED CALCULUS PRACTICE PROBLEMS

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The problems that follow illustrate the methods covered in class. They are typical of the types of problems that will be on the tests.

1. The Real Numbers

Problem 1. State the axioms for the the following operations and relations for the real numbers $\mathbb{R}$: (1) addition $(+)$, (2) multiplication $(\cdot)$, and (3) less than $(<)$.

Problem 2. State the **Least Upper Bound Property** for the real numbers.

Problem 3. Show that for every $x \in \mathbb{R}$ there is an integer M such that $M > x$.

Problem 4. Show that for every $\varepsilon > 0$, there is a positive integer n such that $\frac{1}{n} < \varepsilon$.

Problem 5. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of real numbers. Suppose that $z \in \mathbb{R}$. What does $\lim_{i\to\infty} x_i = z$ mean? We say that $\{x_i\}_{i=1}^{\infty}$ **converges** to z and that the sequence is **convergent**. We say that z is the **limit point** for the sequence.

Problem 6. Suppose that $\{x_i\}_{i=1}^{\infty}$ is a sequence of real numbers such that for all $i < j$, $x_i \leq x_j$. We say that $\{x_i\}_{i=1}^{\infty}$ is **monotone non-decreasing**. Suppose also that the sequence is bounded above. Show that the sequence is convergent. What is the limit?

Problem 7. Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Problem 8. Show that $\lim_{n\to\infty} x^n = 0$ for all $0 < x < 1$. Show this for all $|x| < 1$.

Problem 9. Show that $\sum_{i=0}^{\infty} x^n = \frac{1}{1-x}$ for all $|x| < 1$.

Problem 10. Show that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. If you were trying to determine if the series converges on your calculator, what conclusion might you arrive at? What limit might the series seem to have?

Problem 11. Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. Can you determine the limit?

Problem 12. Show that every nonnegative real number x has a decimal expansion, $x = a_0.a_1, a_2, \cdots$ where $a_0 \in \mathbb{N} \cup \{0\}$ and $a_i \in \{0, 1, 2, \ldots, 9\}$ for all $i > 0$. Are the decimal representations unique? When does a non-negative real number have more than one decimal representation? When does a decimal representation represent a rational number?

2. Sequential Compactness

Problem 13. The **Bolzano-Weierstrass Theorem** states that if $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$, then $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} x_{n_j} = z$ in $\mathbb{R}$. Prove the Bolzano-Weierstrass Theorem.

Problem 14. Let X be a metric space. A subset $A \subset X$ is *closed* provided that for every sequence $\{x_n\}_{n=1}^{\infty}$ in A such that $\lim_{n\to\infty} x_n = z$, then $z \in A$. A subset $A \subset X$ is *sequentially compact* or *compact* provided that every sequence $\{x_n\}_{n=1}^{\infty} \subset A$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} x_{n_j} = z \in A$. Show that $A \subset \mathbb{R}$ is compact if and only if A is closed and bounded.

Problem 15. Show that a closed bounded interval $[a, b]$ is compact. Show that the Cantor Middle Third Set is compact. One description of the **Cantor Set** is $C = \{x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \big| a_n \in \{0, 2\}, n = 1, 2, \ldots\}$.

Problem 16. Show that $A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded.

3. Functions and Continuity

Problem 17. Let X and Y be metric spaces. A function $f : X \to Y$ is *continuous* provided that for every sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} = z \in X$, then $f(x_n) \to f(z)$ as $n \to \infty$ in Y. Show that there is a continuous function $f : C \to [0, 1]$ which is onto, where C is the Cantor Middle Third Set. Show that there is a continuous function $f : C \to [0, 1] \times [0, 1]$. Show that there is a continuous $f : [0, 1] \to [0, 1] \times [0, 1]$. [The first such space-filling curve was discovered by Giuseppe Peano in 1890. David Hilbert described another such function in 1891.]

Problem 18. A set X is *countable* provided that it is finite or that there is a function $f : \mathbb{N} \to X$ which is one-to-one and onto. Show that $\mathbb{N}$ is countable, $\mathbb{Z}$ is countable, $\mathbb{Z}^n$ is countable, and $\mathbb{Q}$ is countable. Show that if X is countable and $A \subset X$, then A is countable.

Problem 19. Show that $[0, 1]$ is uncountable. Show that the Cantor Set C is uncountable.

Problem 20. Show that if X is any set, then there is no function $f : X \to 2^X$ such that f is onto.

Problem 21. Suppose that X and Y are metric spaces and that $A \subset X$ is compact. Suppose that $f : X \to Y$ is continuous. Show that $f(A)$ is compact.

4. Connectedness

Problem 22. Suppose that X is a metric space and that $A \subset X$. We say that A is *disconnected* provided that $A = B_1 \cup B_2$ such that for every sequence $\{x_n\}_{n=1}^{\infty} \subset B_i$ with $x_n \to z$ as $n \to \infty$, then $z \in B_i$ for $i = 1, 2$. We say that $\{B_1, B_2\}$ is a *separation* for A. If A is not disconnected, then we say that A is *connected*. Suppose that X and Y are metric spaces and that $f : X \to Y$ is continuous. Show that if A is connected in X, then $f(A)$ is connected in Y.

Problem 23. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $f([a, b]) \supset [a, b]$. Show that there is an $x \in [a, b]$ such that $f(x) = x$.

Problem 24. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $f([a, b]) \supset [c, d]$. Show that there is an interval $[x, y] \subset [a, b]$ such that $f([x, y]) = [c, d]$.

Problem 25. Show that $A \subset \mathbb{R}$ is connected if and only if A is an interval.

Problem 26. State **Sharkovsky's Theorem**. Prove that if $f : [a, b] \to [a, b]$ is continuous and x_0 is a point having period 3, then for every $n \in \mathbb{N}$, there is a point $x_n \in [a, b]$ such that x_n has period n.

Problem 27. Suppose $f : [a, b] \to [a, b]$ is continuous and x_0 is a point having period 3. How many points of period 5 are there? How many orbits of period 5? How many points of period 29 are there? How many orbits of period 29? How many points of period 229? Note that 29 and 229 are prime numbers. How many points of period 6?

Problem 28. Suppose $f : [a, b] \to [a, b]$ is continuous and x_0 is a point having period 5. Suppose that $\{x_0, f(x_0) = x_1, f(x_1) = x_2, f(x_2) = x_3, f(x_3) = x_4, f(x_4) = x_0\}$ is the orbit with $x_2 < x_3 < x_4 < x_0 < x_1$. What is the **Markov Graph** for this orbit? What is the **Adjacency Matrix** for this orbit? Is there an orbit of period 3? How many orbits of period 3 are there? How many orbits of period 229?

Problem 29. Let $U \subset \mathbb{R}^n$ be a connected open set. Suppose that $x, y \in U$. Show that there is a continuous $f : [0, 1] \to U$ such that $f(0) = x$ and $f(1) = y$.

5. Completeness

Problem 30. Let X be a metric space. Then X is *complete* provided that every Cauchy sequence converges. Show that $\mathbb{R}$ is complete. Show that $\mathbb{R}^n$ is complete. Show that $\mathbb{Q}$ is not complete.

Problem 31. Let X be a metric space. Suppose that $f : X \to X$ is a function. We say that f is a *contraction mapping* provided that there is a $0 < c < 1$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq c \cdot d(x, y)$. Suppose that X is a complete metric space and that $f : X \to X$ is a contraction mapping. Then there is a unique point z such that $f(z) = z$. Furthermore, for every $x_0 \in X$, $\lim_{n\to\infty} f^n(x_0) = z$. This is known as the **Banach Fixed Point Theorem**. It is also known as the **Contraction Mapping Theorem**.

Problem 32. Let X be a complete metric space. Suppose that $\{U_n\}_{n=1}^{\infty}$ is a sequence of dense open sets in X. Then $\cap_{n=1}^{\infty} U_n$ is dense in X. This is known as the **Baire Category Theorem**.

Problem 33. Suppose that X is a complete metric space and that $A \subset X$ is closed. Show that A is a complete metric space.

Problem 34. Suppose that X is a complete metric space and that $\{A_i\}_{i=1}^{\infty}$ is a nested sequence of closed subsets of X such that $\operatorname{diam}(A_n) \to 0$ as $n \to \infty$. Show that $\cap_{i=1}^{\infty} A_i = \{x\}$ for a unique $x \in X$.

Problem 35. Suppose that X is a metric space and that $A \subset X$ is compact. Show that A is also complete.

Problem 36. Suppose that X and Y are complete metric spaces. Show that $X \times Y$ is complete with the metric $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$.

Problem 37. Suppose that X is a metric space and that $A \subset X$ is complete in the metric inherited from X. Show that A is closed in X.

6. Differentiation

Problem 38. Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point x_0, then f is continuous at x_0.

Problem 39. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 and that for some $\varepsilon > 0$, $f(x_0) \geq f(x)$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Show that $f'(x_0) = 0$.

Problem 40. Suppose that X is a compact metric space and that $f : X \to \mathbb{R}$ is continuous. Show that there is an $x_0 \in X$ such that for all $x \in X$, $f(x_0) \geq f(x)$.

Problem 41. Consider $f(x) = x^2 \cdot \sin\left(\frac{1}{x}\right)$ for all $x \neq 0$. Define $f(0) = 0$. Show that $f(x)$ is differentiable for all $x \in \mathbb{R}$ and that $f'(x)$ is not continuous at 0.

Problem 42. The **Mean Value Theorem** states the following. Let $f : [a, b] \to \mathbb{R}$ be continuous with $f(x)$ differentiable for all $a < x < b$. Then there is a c, $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Prove the Mean Value Theorem.

Problem 43. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and that $f(z) = z$ is a fixed point. Suppose that $|f'(z| < 1$. Show that there is an $\varepsilon > 0$ such that for all $x_0 \in (z - \varepsilon, z + \varepsilon)$, $f^n(x_0) \to z$ as $n \to \infty$. Such a fixed point z is called an *attracting fixed point*. What happens at z if $|f'(z)| > 1$? Such a fixed point is called a *repelling fixed point*.

Problem 44. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and suppose that $f(z) = 0$. Suppose also that $f'(z) \neq 0$. Define $g(x) = x - \frac{f(x)}{f'(x)}$. Show that z is an attracting fixed point for g.

Problem 45. Determine numerical solutions of the following equations.

$$\cos(x) = x$$

$$x^{15} + 3x^4 - 4x^3 + x = 5$$

Problem 46. Let $f_\mu : [0, 1] \to [0, 1]$ be defined by $f_\mu(x) = \mu \cdot x \cdot (1 - x)$ for $0 \le \mu \le 4$. Suppose that for some n and some $x_0 \in [0, 1]$, the derivative of $f_\mu^n(x)$ is zero at $x = x_0$. Show that for some $0 \le k < n$, $f^k(x) = \frac{1}{2}$.

Problem 47. Consider the function $f_\mu(x) = \mu \cdot x \cdot (1 - x)$. Determine the values of μ such that $f_\mu^3\left(\frac{1}{2}\right) = \frac{1}{2}$. Show that for one of these values of $\mu \in [0, 4]$, $\frac{1}{2}$ is a periodic point of period three. Show that for this value of μ, the derivative $\frac{d}{dx} f_\mu^3(x)\big|_{x=\frac{1}{2}} = 0$.

Problem 48. Consider the function $f_\mu(x) = \mu \cdot x \cdot (1 - x)$. Determine the values of μ such that $f_\mu^5\left(\frac{1}{2}\right) = \frac{1}{2}$. Show that for three of these values of $\mu \in [0, 4]$, $\frac{1}{2}$ is a periodic point of period five. Show that for these value of μ, the derivative $\frac{d}{dx} f_\mu^5(x)\big|_{x=\frac{1}{2}} = 0$. Thus, these are attracting periodic orbits of period five.

Problem 49. Define $f(x) = \exp\left(-\frac{1}{x^2}\right)$ for $x \neq 0$ and $f(0) = 0$. Show that for every $n \ge 0$, $\frac{d^n f}{dx^n}\big|_{x=0} = 0$. What is the Taylor Series for this $f(x)$ centered at $a = 0$?

Problem 50. The **Taylor Remainder Theorem** states the following. Suppose that $f(x)$ is $(N + 1)$ times differentiable on $[a, b]$ with $a < x_0 < b$. Let $a < x < b$, then there is a point ξ between x_0 and x such that the following holds.

$$
\begin{aligned}
f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''}{2}(x-x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N + \frac{f^{N+1}(\xi)}{(N+1)!}(x-x_0)^{N+1} \\
&= \sum_{n=0}^{N} \frac{f^{(n)}}{n!}(x-x_0)^n + \frac{f^{N+1}(\xi)}{(N+1)!}(x-x_0)^{N+1}
\end{aligned}
$$

Prove the Taylor Remainder Theorem.