MAA 4211 – James Keesling – Quiz 2

1. State and prove the Banach Fixed Point Theorem.

2. Show that $\mathbb{R}$ is a complete metric space. Show that $\mathbb{R}^n$ is a complete metric space.

3. Suppose that X is a complete metric space and that $A \subset X$ is a closed subset. Show that A is a complete metric space.

4. Suppose that X is a complete metric space and that $\{A_i\}_{i=1}^{\infty}$ is a nested sequence of closed subsets of X such that $\text{diam}(A_i) \to 0$ as $n \to \infty$. Show that $\cap_{i=1}^{\infty} A_i = \{x\}$ for some unique $x \in X$.

5. Suppose that X is a metric space. Suppose that $A \subset X$ is compact. Show that A is complete.

6. Suppose that X is a metric space and suppose that $A \subset X$ is complete. Show that there is a sequence of open sets in X, $\{U_i\}_{i=1}^{\infty}$ such that for all i, $U_i \supset A$ with $\cap_{i=1}^{\infty} U_i = A$. We say that A is a G_δ.

7. Suppose that X and Y are complete metric spaces. Show that $X \times Y$ is a complete with the metric $d((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2)$.

8. Let $U \subset \mathbb{R}^n$ be a connected open subset of $\mathbb{R}^n$. Suppose that $x, y \in U$. Show that there is a continuous function $f:[0,1] \to U$ such that $f(0) = x$ and $f(1) = y$.