MAA 4211 QUIZ 4 FALL 2017 - JAMES KEESLING

Problem 1. Show that if $f:\mathbb{R}\to\mathbb{R}$ is differentiable at a point x_0, then f is continuous at x_0.

Problem 2. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is differentiable at x_0 and that for some $\varepsilon>0$, $f(x_0)\ge f(x)$ for all $x\in(x_0-\varepsilon,x_0+\varepsilon)$. Show that $f'(x_0)=0$.

Problem 3. Suppose that X is a compact metric space and that $f:X\to\mathbb{R}$ is continuous. Show that there is an $x_0\in X$ such that for all $x\in X$, $f(x_0)\ge f(x)$.

Problem 4. Consider $f(x)=x^2\cdot\sin\left(\frac{1}{x}\right)$ for all $x\neq 0$. Define $f(0)=0$. Show that $f(x)$ is differentiable for all $x\in\mathbb{R}$ and that $f'(x)$ is not continuous at 0.

Problem 5. The **Mean Value Theorem** states the following. Let $f:[a,b]\to\mathbb{R}$ be continuous with $f(x)$ differentiable for all $a<x<b$. Then there is a c, $a<c<b$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Prove the Mean Value Theorem.

Problem 6. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and that $f(z)=z$ is a fixed point. Suppose that $|f'(z)|<1$. Show that there is an $\varepsilon>0$ such that for all $x_0\in(z-\varepsilon,z+\varepsilon)$, $f^n(x_0)\to z$ as $n\to\infty$. Such a fixed point z is called an *attracting fixed point*. What happens at z if $|f'(z)|>1$? Such a fixed point is called a *repelling fixed point*.

Problem 7. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and suppose that $f(z)=0$. Suppose also that $f'(z)\neq 0$. Define $g(x)=x-\frac{f(x)}{f'(x)}$. Show that z is an attracting fixed point for g.

Problem 8. Determine numerical solutions of the following equations.

$$\cos(x)=x$$

$$x^{15}+3x^4-4x^3+x=5$$

Problem 9. Let $f_\mu:[0,1]\to[0,1]$ be defined by $f_\mu(x)=\mu\cdot x\cdot(1-x)$ for $0\le\mu\le 4$. Suppose that for some n and some $x_0\in[0,1]$, the derivative of $f_\mu^n(x)$ is zero at $x=x_0$. Show that for some $0\le k<n$, $f^k(x)=\frac{1}{2}$.

Problem 10. Consider the function $f_\mu(x) = \mu \cdot x \cdot (1-x)$. Determine the values of μ such that $f_\mu^3\left(\frac{1}{2}\right) = \frac{1}{2}$. Show that for one of these values of $\mu \in [0,4]$, $\frac{1}{2}$ is a periodic point of period three. Show that for this value of μ, the derivative $\frac{d}{dx} f_\mu^3(x)\Big|_{x=\frac{1}{2}} = 0$.

Problem 11. Consider the function $f_\mu(x) = \mu \cdot x \cdot (1-x)$. Determine the values of μ such that $f_\mu^5\left(\frac{1}{2}\right) = \frac{1}{2}$. Show that for three of these values of $\mu \in [0,4]$, $\frac{1}{2}$ is a periodic point of period five. Show that for these value of μ, the derivative $\frac{d}{dx} f_\mu^5(x)\Big|_{x=\frac{1}{2}} = 0$. Thus, these are attracting periodic orbits of period five.

Problem 12. Define $f(x) = \exp\left(-\frac{1}{x^2}\right)$ for $x \neq 0$ and $f(0) = 0$. Show that for every $n \geq 0$, $\frac{d^n f}{dx^n}\Big|_{x=0} = 0$. What is the Taylor Series for this $f(x)$ centered at $a = 0$?

Problem 13. The **Taylor Remainder Theorem** states the following. Suppose that $f(x)$ is $(N+1)$ times differentiable on $[a,b]$ with $a < x_0 < b$. Let $a < x < b$, then there is a point ξ between x_0 and x such that the following holds.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''}{2}(x-x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N + \frac{f^{N+1}(\xi)}{(N+1)!}(x-x_0)^{N+1} \\ &= \sum_{n=0}^{N} \frac{f^{(n)}}{n!}(x-x_0)^n + \frac{f^{N+1}(\xi)}{(N+1)!}(x-x_0)^{N+1} \end{aligned}$$

Prove the Taylor Remainder Theorem.