Problem 1. Show that a closed bounded interval \([a,b]\) is compact. Show that the Cantor Middle Third Set is compact. One description of the Cantor Set is \(C = \{x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} | a_n \in \{0,2\}, n = 1,2,\ldots\}\).

Problem 2. Show that \(A \subset \mathbb{R}\) is compact if and only if \(A\) is closed and bounded.

Problem 3. A set \(X\) is countable provided that it is finite or that there is a function \(f : \mathbb{N} \rightarrow X\) which is one-to-one and onto. Show that \(\mathbb{N}\) is countable, \(\mathbb{Z}\) is countable, \(\mathbb{Z}^n\) is countable, and \(\mathbb{Q}\) is countable. Show that if \(X\) is countable and \(A \subset X\), then \(A\) is countable.

Problem 4. Show that \([0,1]\) is uncountable. Show that the Cantor Set \(C\) is uncountable.

Problem 5. Show that if \(X\) is any set, then there is no function \(f : X \rightarrow 2^X\) such that \(f\) is onto.

Problem 6. Let \(U \subset \mathbb{R}^n\) be a connected open set. Suppose that \(x,y \in U\). Show that there is a continuous \(f : [0,1] \rightarrow U\) such that \(f(0) = x\) and \(f(1) = y\).

Problem 7. Let \(X\) be a metric space. Then \(X\) is complete provided that every Cauchy sequence converges. Show that \(\mathbb{R}\) is complete. Show that \(\mathbb{R}^n\) is complete. Show that \(\mathbb{Q}\) is not complete.

Problem 8. Let \(X\) be a metric space. Suppose that \(f : X \rightarrow X\) is a function. We say that \(f\) is a contraction mapping provided that there is a \(0 < c < 1\) such that for all \(x,y \in X\), \(d(f(x), f(y)) \leq c \cdot d(x,y)\). Suppose that \(X\) is a complete metric space and that \(f : X \rightarrow X\) is a contraction mapping. Then there is a unique point \(z\) such that \(f(z) = z\). Furthermore, for every \(x_0 \in X\), \(\lim_{n \rightarrow \infty} f^n(x_0) = z\). This is known as the Banach Fixed Point Theorem. It is also known as the Contraction Mapping Theorem.

Problem 9. Suppose that \(X\) is a complete metric space and that \(A \subset X\) is closed. Show that \(A\) is a complete metric space.

Problem 10. Suppose that \(X\) is a metric space and that \(A \subset X\) is compact. Show that \(A\) is also complete.

Problem 11. Suppose that \(X\) and \(Y\) are complete metric spaces. Show that \(X \times Y\) is complete with the metric \(d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)\).

Problem 12. Suppose that \(X\) is a metric space and that \(A \subset X\) is complete in the metric inherited from \(X\). Show that \(A\) is closed in \(X\).
Problem 13. Determine numerical solutions of the following equations.

\[
\cos(x) = x
\]

\[
x^{15} + 3x^4 - 4x^3 + x = 5
\]

Problem 14. State and prove Cauchy’s Mean Value Theorem.

Problem 15. State and prove the Mean Value Theorem.

Problem 16. Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous. Show that there is a \( c \in [a, b] \) such that \( f(c) \geq f(x) \) for all \( x \in [a, b] \).

Problem 17. Let \( f : A \to \mathbb{R} \) be a function. We say that \( f \) is Uniformly Continuous provided that for every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that for every \( x \) and \( y \) in \( A \), if \( d(x, y) < \delta \), then \( d(f(x), f(y)) < \epsilon \). Show that if \( A \) is compact and \( f : A \to \mathbb{R} \) is continuous, then \( f : A \to \mathbb{R} \) is uniformly continuous.

Problem 18. Show that if \( f : \mathbb{R} \to \mathbb{R} \) is differentiable at a point \( x_0 \), then \( f \) is continuous at \( x_0 \).

Problem 19. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is differentiable at \( x_0 \) and that for some \( \epsilon > 0 \), \( f(x_0) \geq f(x) \) for all \( x \in (x_0 - \epsilon, x_0 + \epsilon) \). Show that \( f'(x_0) = 0 \).

Problem 20. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is continuously differentiable and that \( f(z) = z \) is a fixed point. Suppose that \( |f'(z)| < 1 \). Show that there is an \( \epsilon > 0 \) such that for all \( x_0 \in (z - \epsilon, z + \epsilon) \), \( f^n(x_0) \to z \) as \( n \to \infty \). Such a fixed point \( z \) is called an attracting fixed point. What happens at \( z \) if \( |f'(z)| > 1 \)? Such a fixed point is called a repelling fixed point.