MAA 4211 QUIZ 5 FALL 2017 - JAMES KEESLING

Problem 1. Show that a closed bounded interval $[a, b]$ is compact. Show that the Cantor Middle Third Set is compact. One description of the **Cantor Set** is $C = \{x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \big| a_n \in \{0, 2\}, n = 1, 2, \ldots\}$.

Problem 2. Show that $A \subset \mathbb{R}$ is compact if and only if A is closed and bounded.

Problem 3. A set X is **countable** provided that it is finite or that there is a function $f : \mathbb{N} \to X$ which is one-to-one and onto. Show that $\mathbb{N}$ is countable, $\mathbb{Z}$ is countable, $\mathbb{Z}^n$ is countable, and $\mathbb{Q}$ is countable. Show that if X is countable and $A \subset X$, then A is countable.

Problem 4. Show that $[0, 1]$ is uncountable. Show that the Cantor Set C is uncountable.

Problem 5. Show that if X is any set, then there is no function $f : X \to 2^X$ such that f is onto.

Problem 6. Let $U \subset \mathbb{R}^n$ be a connected open set. Suppose that $x, y \in U$. Show that there is a continuous $f : [0, 1] \to U$ such that $f(0) = x$ and $f(1) = y$.

Problem 7. Let X be a metric space. Then X is *complete* provided that every Cauchy sequence converges. Show that $\mathbb{R}$ is complete. Show that $\mathbb{R}^n$ is complete. Show that $\mathbb{Q}$ is not complete.

Problem 8. Let X be a metric space. Suppose that $f : X \to X$ is a function. We say that f is a *contraction mapping* provided that there is a $0 < c < 1$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq c \cdot d(x, y)$. Suppose that X is a complete metric space and that $f : X \to X$ is a contraction mapping. Then there is a unique point z such that $f(z) = z$. Furthermore, for every $x_0 \in X$, $\lim_{n\to\infty} f^n(x_0) = z$. This is known as the **Banach Fixed Point Theorem**. It is also known as the **Contraction Mapping Theorem**.

Problem 9. Suppose that X is a complete metric space and that $A \subset X$ is closed. Show that A is a complete metric space.

Problem 10. Suppose that X is a metric space and that $A \subset X$ is compact. Show that A is also complete.

Problem 11. Suppose that X and Y are complete metric spaces. Show that $X \times Y$ is complete with the metric $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$.

Problem 12. Suppose that X is a metric space and that $A \subset X$ is complete in the metric inherited from X. Show that A is closed in X.

Problem 13. Determine numerical solutions of the following equations.

$$\cos(x) = x$$

$$x^{15} + 3x^4 - 4x^3 + x = 5$$

Problem 14. State and prove **Cauchy's Mean Value Theorem**.

Problem 15. State and prove the **Mean Value Theorem**.

Problem 16. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Show that there is a $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$

Problem 17. Let $f : A \to \mathbb{R}$ be a function. We say that f is **Uniformly Continuous** provided that for every $\epsilon > 0$, there is a $\delta > 0$ such that for every x and y in A, if $d(x, y) < \delta$, then $d(f(x), f(y) < \epsilon$. Show that if A is compact and $f : A \to \mathbb{R}$ is continuous, then $f : A \to \mathbb{R}$ is uniformly continuous.

Problem 18. Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point x_0, then f is continuous at x_0.

Problem 19. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 and that for some $\varepsilon > 0$, $f(x_0) \geq f(x)$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Show that $f'(x_0) = 0$.

Problem 20. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and that $f(z) = z$ is a fixed point. Suppose that $|f'(z)| < 1$. Show that there is an $\varepsilon > 0$ such that for all $x_0 \in (z - \varepsilon, z + \varepsilon)$, $f^n(x_0) \to z$ as $n \to \infty$. Such a fixed point z is called an *attracting fixed point*. What happens at z if $|f'(z)| > 1$? Such a fixed point is called a *repelling fixed point*.