Determining a Formula for $\sum_{i=0}^{n} i^k$

Let $x_0, x_1, x_2, \ldots$ be a sequence. Define $X_n = \sum_{i=0}^{n} x_i$ be the sum of the sequence up to the nth term. Define $\Delta_i(\{x_j\}) = x_i - x_{i-1}$ for $i = 1,2,\ldots$. Notice that

$\Delta_i(\{X_n\}) = X_i - X_{i-1} = \sum_{j=0}^{i} x_j - \sum_{j=0}^{i-1} x_j = x_i$ for $i = 1,2,\ldots$.

Theorem. Suppose that $X_n = \sum_{i=0}^{n} x_i$ and that Y_n is any sequence such that (1) $X_0 = Y_0$ and (2) $\Delta_i(\{X_n\}) = \Delta_i(\{Y_n\})$ for all $i = 1,2,\ldots$. Then $X_n \equiv Y_n$ for all $n = 0,1,2,\ldots$.

Proof. Clearly, $X_0 = Y_0$ by (1) of the assumptions. It is also clear that $\Delta_1(\{X_n\}) = x_1 = \Delta_1(\{Y_n\}) = Y_1 - Y_0 = Y_1 - X_0 = Y_1 - x_0$ and thus that $Y_1 = x_0 + x_1 = X_1$. Similarly, $\Delta_2(\{X_n\}) = x_2 = \Delta_2(\{Y_n\}) = Y_2 - Y_1 = Y_2 - X_1 = Y_2 - (x_0 + x_1)$ and thus, $Y_2 = x_0 + x_1 + x_2 = X_2$. It should be clear that we can continue this process inductively to show that $Y_n = X_n$ for all $n = 0,1,2,\ldots$

Example. Find a formula for $X_n = \sum_{i=0}^{n} i^2$.

Solution. Let $Y_n = a \cdot n^3 + b \cdot n^2 + c \cdot n + d$. We want to adjust the coefficients $\{a,b,c,d\}$ so that $X_0 = Y_0$ and $\Delta_i(\{X_n\}) = \Delta_i(\{Y_n\})$ for all $i = 1,2,\ldots$. Then we can conclude from the above theorem that $X_n \equiv Y_n$ for all $n = 0,1,2,\ldots$. Now $X_0 = 0 = Y_0 = d$ gives us that $d = 0$. Thus, we can simplify $Y_n = a \cdot n^3 + b \cdot n^2 + c \cdot n$. Now we compute:

$\Delta_n(\{X_i\}) = n^2$

$\Delta_n(\{Y_i\}) = Y_n - Y_{n-1} = a \cdot n^3 + b \cdot n^2 + c \cdot n - (a \cdot (n-1)^3 + b \cdot (n-1)^2 + c \cdot (n-1))$

This gives us the equation: $\Delta_n(\{X_i\}) = n^2 = 3a \cdot n^2 + (2b - 3a)n + (a - b + c) = \Delta_n(\{Y_i\})$

For this equality to hold we must have that

$3a = 1$
$2b - 3a = 0$
$a - b + c = 0$

These equations can be solved to get $a = \frac{1}{3}$, $b = \frac{1}{2}$, and $c = \frac{1}{6}$. Thus we can conclude that: $\sum_{i=0}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ for all n.

Exercise. Find a formula for $X_n = \sum_{i=0}^{n} i^3$.