ADVANCED CALCULUS PRACTICE PROBLEMS

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The problems that follow illustrate the methods covered in class. They are typical of the types of problems that will be on the tests.

1. Riemann Integration

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. State the definition of the **derivative** of f at a point $a \in \mathbb{R}$.

Problem 2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. State when the **Riemann integral** of $f(x)$ over $[a, b]$, $\int_a^b f(x)dx$, exists. What is the value of $\int_a^b f(x)dx$ when it does exist.

Problem 3. Suppose that $f : [a, b] \to \mathbb{R}$ is monotone. Show that $\int_a^b f(x)dx$ exists.

Problem 4. Let $C = \left\{x \in [0, 1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, a_n \in \{0, 2\}\right\}$ be the **Cantor set**. Let $f : C \to [0, 1]$ be defined by $fx) = \sum_{n=1}^{\infty} \frac{a_n/2}{2^n}$ where $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ is an element of the Cantor set. This is called the **Cantor ternary function**. This function can be extended so that $f : [0, 1] \to [0, 1]$ by making f be constant on the intervals that are complementary to the Cantor set. Compute the Riemann integral $\int_0^1 f(x)dx$ of this function $f(x)$.

Problem 5. Let $\{r_i\}_{i=1}^{\infty}$ be an enumeration of the rational numbers in $(0, 1)$. Define $f : [0, 1] \to [0, 1]$ by the formula $f(x) = \sum_{r_i<x} \frac{1}{2^i}$ for $0 < x \le 1$ and $f(0) = 0$. Show that f is an increasing function. Determine the value of $\int_0^1 f(x)dx$.

Problem 6. Determine a formula for $\sum_{i=1}^{n} i^2$. Use this to determine $\int_0^1 x^2 dx$ using the Riemann sum definition of the integral.

Problem 7. Suppose that $f(x)$ is piecewise monotone on $[a, b]$. By that we mean that $[a, b] = [x_0 = a, x_1] \cup [x_1, x_2] \cup \cdots [x_{n-1}, x_n = b]$ with $f(x)$ monotone on $[x_i, x_{i+1}]$ for $0 \le i < n$. Show that the Riemann integral for $f(x)$, $\int_a^b f(x)dx$, exists.

Problem 8. State and prove the **Fundamental Theorem of Calculus**.

Problem 9. Suppose that $f(x)$ is continuous on $[a,b]$. Show that the Riemann integral for $f(x)$, $\int_a^b f(x)dx$, exists.

Problem 10. Explain how Romberg integration works. Be able to use the TI-Nspire CX CAS program to determine the Romberg estimate of integral.

2. Definition of $\ln(x)$ and $\exp(x)$

Problem 11. Define $\ln(x) = \int_1^x \frac{1}{t}dt$. Show that $\frac{d\ln(x)}{dx} \equiv \frac{1}{x}$. Show that $\ln(x)$ is strictly monotone increasing. Show that $\ln(x \cdot y) = \ln(x) + \ln(y)$ for all $x, y > 0$. Show that $\lim_{x\to\infty} \ln(x) = \infty$ and that $\lim_{x\to 0^+} \ln(x) = -\infty$

Problem 12. Define $\exp(x) = y$ where $\ln(y) = x$. Show that $\exp(x+y) = \exp(x)\cdot\exp(y)$, $\lim_{x\to-\infty} = 0$, and $\lim_{x\to\infty} = \infty$. Show also that $\frac{d\exp(x)}{dx} \equiv \exp(x)$.

Problem 13. Let $a > 0$. Define $a^b = \exp(b \cdot \ln(a))$. Calculate the following using this definition.

$$\frac{d}{dx}a^x$$

$$\frac{d}{dx}x^x$$

$$\lim_{x\to+\infty}(1+1/x)^x$$

Problem 16. Define $e > 0$ by $\ln(e) = 1$. Show that $\exp(x) \equiv e^x$ for this e.

Problem 17. Define $\log_b(x) = y$ such that $b^y = x$. Show that $\log_b(x) \equiv \frac{\ln(x)}{\ln(b)}$.

3. Pointwise and uniform convergence

Problem 18. Define *pointwise convergence* and *uniform convergence*. Show that $\{f_n(x) = x^n\}_{n=1}^{\infty}$ converges pointwise to the following function.

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

Show that this convergence is not uniform.

Problem 19. Suppose that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f(x)$ on $[a, b]$. Suppose that $f_n(x)$ is Riemann integrable for all n. Suppose also that $f(x)$ is Riemann integrable. Show that

$$\lim_{n\to\infty} \int_a^b f_n(x) = \int_a^b f(x)dx.$$

Problem 20. Suppose that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f(x)$ on $[a, b]$. Suppose that for each n $f_n(x)$ is continuous on $[a, b]$. Show that $f(x)$ is continuous on $[a, b]$.

4. The geometric series

Problem 21. Show that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for all $|x| < 1$. Let $0 < \epsilon < 1$. Show that the convergence $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ is uniform for $|x| < 1 - \epsilon$.

Problem 22. Prove the following holds for al $|x| < 1$.

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot x^n}{n}$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2n + 1}$$

5. Derivatives

Problem 23. Suppose that $f(x)$ is differentiable on $[a, b]$ and that $f'(x) > 0$ for all $x \in (a, b)$. Show that $f(x)$ is strictly increasing on $[a, b]$. Suppose that $f(x) \equiv 0$ on $[a, b]$. Show that there is a constant C such that $f(x) \equiv C$ on $[a, b]$

Problem 24. Define $f(x)$ in the following way.

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{x}{2} + x^2 \cdot \sin\left(\frac{1}{x}\right) & x \neq 0 \end{cases}$$

Show that $f'(0) > 0$, but that $f(x)$ is not increasing on any interval containing 0.

Problem 25. Suppose that $f_n(x)$ converges uniformly to $f(x)$ on $[a, b]$. Is it true that $f'_n(x)$ converges to $f'(x)$? Prove this if it does. Give a counterexample if it does not.

Problem 26. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $f'(x)$ is bounded on $\mathbb{R}$. Show that $f(x)$ is uniformly continuous on $\mathbb{R}$.

6. Some examples and applications of integration

Problem 27. Suppose that $f(x)$ is continuous on $[a, b]$. Show that $f(x)$ is uniformly continuous on $[a, b]$.

Problem 28. Let $f(x)$ be defined as below.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

This is known as the **Dirichlet function**. Show that $f(x)$ is not Riemann integrable on $[0, 1]$.

Problem 29. Let $f(x)$ be define as below.

$$f(x) = \begin{cases} \frac{1}{n} & x = \frac{p}{n} \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

This is known as the **popcorn function** or **Thomae's function**. Show that $f(x)$ is Riemann integrable on $[0, 1]$. What is $\int_0^1 f(x)dx$?

Problem 30. State **Cavalieri's Principle**. Determine the volume of a sphere using this principle. Determine the volume of a solid torus using the method.

Problem 31. Use Cavalieri's method to determine the area in an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Problem 32. Give the definition of the *centroid* of an area A. What is the centroid of the area in a circle? What is the centroid of the area in an ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$? What is the centroid of a half-circle?

Problem 33. Give the definition of the *centroid* of an arc L. What is the centroid of the circumference of a circle? What is the centroid of the circumference of half a circle?

Problem 34. Determine the volume of a cone whose base has area A and whose vertex is distance h from the plane of the base.

Problem 35. State **Pappus' Theorem**. Prove Pappus' Theorem. Use Pappus' Theorem to determine the volume of a torus determined by rotating a circle of radius b about the $y-$axis where the center of the circle is on the $x-$axis at a distance a from the origin..

Problem 36. Use Pappus' Theorem to determine the surface area of a torus given by rotating a circle of radius b about the $y-$axis where the center of the circle is on the $x-$axis at a distance a from the origin.

Problem 37. Give the definition of the centroid of a solid figure V. Determine the centroid of a right circular cone. Determine the centroid of a cone with base A whose vertex is a distance h from the plane of the base.

Problem 38. Let V be a right circular cone with height h and radius at the base r. Suppose that the cone has density d relative to the density of water with $0 < d < 1$. Determine when the cone will float stably with the vertex downward. Similarly, let V be a solid hemisphere of radius r and density $\delta > 0$ less than that of water. Show that it will float stably when its flat surface is parallel to the surface of the water.

Problem 39. Determine the arclength of the graph of $y = x^2$ over $[0, a]$. Determine the centroid of this arc. Determine the surface area rotating the figure around the $x-$axis and around the $y-$axis.

Problem 40. Determine the arclength of a catenary having the following equation

$$y = a \cdot \cosh\left(\frac{x}{a}\right) = a \cdot \frac{\left(\exp\left(\frac{x}{a}\right) + \exp\left(-\frac{x}{a}\right)\right)}{2}$$

over the interval $[0, b]$.

Problem 41. Determine the circumference of a circle of radius a using the arclength integral.

Problem 42. Determine the centroid of the perimeter of the upper half of a circle given by $x^2 + y^2 = a^2$. Determine the centroid of the upper half of the area of a circle having this equation.

7. Power series

Problem 43. Show that the following power series converge to the given functions on the interval $(-1,1)$. What happens at $x=1$ in each of these cases?

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \qquad \arctan(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

Problem 44. Show that $\lim_{n\to\infty}\sqrt[n]{n}=1$. Show that $\lim_{n\to\infty}\sqrt[n]{n+1}=1$. Use this to show that

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}(n+1)x^n$$

for all $-1<x<1$.

Problem 45. Let $f(x)=\sum_{n=0}^{\infty}a_nx^n$. Show that the sum converges for all $|x|<R$ where

$$\limsup_{n\to\infty}\sqrt[n]{|a_n|}=\alpha=\frac{1}{R}.$$

We say that R is the **radius of convergence** for the power series.

Problem 46. Suppose that R is the radius of convergence for the power series $f(x)=\sum_{n=0}^{\infty}a_nx^n$. Show that if $g(x)=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$, then the radius of convergence for $g(x)$ is also R. Show that $\frac{d}{dx}f(x)=g(x)$ for all $-R<x<R$.

Problem 47. Suppose that R is the radius of convergence for the power series $f(x)=\sum_{n=0}^{\infty}a_nx^n$. Show that $f(x)$ is infinitely differentiable for all $-R<x<R$. Show that $a_n=\frac{f^{(n)}(x)|_{x=0}}{n!}$ for all $n=0,1,2,\ldots$.

8. Differential equations

Problem 48. Consider a differential equation of the form

$$\frac{dx}{dt}=f(t,x) \qquad x(t_0)=x_0.$$

This is equivalent to the following equation.

$$x(t)=x_0+\int_{\tau=t_0}^{t}f(\tau,x(\tau))d\tau$$

The solution will be a function $x(t)$ such that $x(t_0) = x_0$ and which satisfies the equaiton $\frac{dx}{dt} = f(t, x(t))$ identically. Use Picard Iteration to approximate the solution to the following differential equations. Use five iterations.

$$\frac{dx}{dt} = t \cdot x \qquad x(0) = 2$$

$$\frac{dx}{dt} = x \cdot \sin(t) \qquad x(0) = -1$$

Problem 49. Determine a polynomial of degree 5 that approximates a solution to the following differential equation on a interval centered at $t = 1$.

$$\frac{dx}{dt} = x \cdot \sin(t) \qquad x(1) = 2$$

For the same differential equation approximate the solution at a set of grid points using the Taylor Method of degree 5 and using stepsize $h = 1/10$ and $n = 10$. This estimates the solution at $t = 0, \frac{1}{10}, \frac{2}{10}, \cdots, 1$. How accurate is this numerical estimate of the solution?

Problem 50. State and prove the **Contraction Mapping Theorem**. This is also known as the **Banach Fixed Point Theorem**.

Problem 51. Show that the following differential equation does not have a unique solution.

$$\frac{dx}{dt} = \sqrt{x} \qquad x(0) = 0$$

Problem 52. Consider the linear system of differential equations given by the following equations

$$\frac{dx}{dt} = Ax \qquad\qquad x(0) = C$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

As shown in class, the solution of this linear system of equations is

$$x(t) = \exp(t \cdot A) \cdot C.$$

Solve this differential equation for the following conditions.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Solve on $[0,1]$ with $h = 1/20$ and $n = 20$.

9. Function Spaces

Problem 53. Let X be a compact metric space and let $C(X,\mathbb{R})$ be the set of all continuous functions $f : X \to \mathbb{R}$. Define a metric $d(f,g)$ on $C(X,\mathbb{R})$ by $d(f,g) = \max_{x\in X} |f(x)-g(x)|$. Show that this satisfies the properties of being a metric. Show that $C(X,\mathbb{R})$ is complete in this metric.

Problem 54. Consider the differential equation

$$\frac{dx}{dt} = f(t,x) \qquad x(t_0) = x_0.$$

Assume that $f(t,x)$ is continuous in a rectangle containing (t_0,x_0). Assume also that $f(t,x)$ is **Lipschitz** in x in that rectangle. Show that there is a positive $\varepsilon > 0$ such that there is a unique solution $x(t)$ to the differential equation on $[t_0-\varepsilon, t_0+\varepsilon]$. The proof should use **Picard Iteration** and the **Contraction Mapping Theorem** together with the fact that $C(X,\mathbb{R})$ is a complete metric space.

Problem 55. State and prove the **Baire Category Theorem**.

Problem 56. Use the **Baire Category Theorem** together with the completeness of $C(I,\mathbb{R})$ to show that there is a dense G_δ, $B \subset C(I,\mathbb{R})$ such that every function $f \in B$ not differentiable at any point.