The problems that follow illustrate the methods covered in class. They are typical of
the types of problems that will be on the tests.

1. The Real Numbers

Problem 1. State the axioms for the the following operations and relations for the real
numbers \( \mathbb{R} \): (1) addition (+), (2) multiplication (\( \cdot \)), and (3) less than (\( < \)).

Problem 2. State the Least Upper Bound Property for the real numbers.

Problem 3. Show that for every \( x \in \mathbb{R} \) there is an integer \( M \) such that \( M > x \).

Problem 4. Show that for every \( \varepsilon > 0 \), there is a positive integer \( n \) such that \( \frac{1}{n} < \varepsilon \).

Problem 5. Let \( \{x_i\}_{i=1}^{\infty} \) be a sequence of real numbers. Suppose that \( z \in \mathbb{R} \). What does
\( \lim_{i \to \infty} x_i = z \) mean?

Problem 6. Let \( \{x_i\}_{i=1}^{\infty} \) be a sequence of real numbers. Suppose that \( z \in \mathbb{R} \). What
does \( \lim_{i \to \infty} x_i = z \) mean? We say that \( \{x_i\}_{i=1}^{\infty} \) converges to \( z \) and that the sequence is convergent. We say that \( z \) is the limit point for the sequence.

Problem 7. Suppose that \( \{x_i\}_{i=1}^{\infty} \) is a sequence of real numbers such that for all \( i < j \),
\( x_i \leq x_j \). We say that \( \{x_i\}_{i=1}^{\infty} \) is monotone non-decreasing. Suppose also that the
sequence is bounded above. Show that the sequence is convergent. What is the limit?

Problem 8. Show that \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

Problem 9. Show that \( \lim_{n \to \infty} x^n = 0 \) for all \( 0 < x < 1 \). Show this for all \( |x| < 1 \).

Problem 10. Show that \( \sum_{i=0}^{\infty} x^n = \frac{1}{1-x} \) for all \( |x| < 1 \).
Problem 11. Show that \( \sum_{n=1}^{\infty} \frac{1}{n} = \infty \). If you were trying to determine if the series converges on your calculator, what conclusion might you arrive at? What limit might the series seem to have?

Problem 12. Show that \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \) converges. Can you determine the limit?

Problem 13. Show that every nonnegative real number \( x \) has a decimal expansion, \( x = a_0.a_1a_2 \cdots \) where \( a_0 \in \mathbb{N} \cup \{0\} \) and \( a_i \in \{0,1,2,\ldots,9\} \) for all \( i > 0 \). Are the decimal representations unique? When does a non-negative real number have more than one decimal representation? When does a decimal representation represent a rational number?

2. Sequential Compactness

Problem 14. The Bolzano-Weierstrass Theorem states that if \( \{x_n\}_{n=1}^{\infty} \) is a bounded sequence in \( \mathbb{R} \), then \( \{x_n\}_{n=1}^{\infty} \) has a subsequence \( \{x_{n_j}\}_{j=1}^{\infty} \) such that \( \lim_{j \to \infty} x_{n_j} = z \) in \( \mathbb{R} \). Prove the Bolzano-Weierstrass Theorem.

Problem 15. Let \( X \) be a metric space. A subset \( A \subset X \) is closed provided that for every sequence \( \{x_n\}_{n=1}^{\infty} \) in \( A \) such that \( \lim_{n \to \infty} x_n = z \), then \( z \in A \). A subset \( A \subset X \) is sequentially compact or compact provided that every sequence \( \{x_n\}_{n=1}^{\infty} \subset A \) has a subsequence \( \{x_{n_j}\}_{j=1}^{\infty} \) such that \( \lim_{j \to \infty} x_{n_j} = z \in A \). Show that \( A \subset \mathbb{R} \) is compact if and only if \( A \) is closed and bounded.

Problem 16. Show that a closed bounded interval \([a,b]\) is compact. Show that the Cantor Middle Third Set is compact. One description of the Cantor Set is \( C = \{x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} | a_n \in \{0,2\}, n = 1,2,\ldots\} \).

Problem 17. Show that \( A \subset \mathbb{R}^n \) is compact if and only if \( A \) is closed and bounded.
Problem 18. Let $X$ and $Y$ be metric spaces. A function $f : X \to Y$ is continuous provided that for every sequence $\{x_n\}_{n=1}^\infty$ in $X$ such that $\lim_{n \to \infty} x_n = z \in X$, then $f(x_n) \to f(z)$ as $n \to \infty$ in $Y$. Show that there is a continuous function $f : C \to [0, 1]$ which is onto, where $C$ is the Cantor Middle Third Set. Show that there is a continuous function $f : [0, 1] \to [0, 1] \times [0, 1]$. [The first such space-filling curve was discovered by Giuseppe Peano in 1890. David Hilbert described another such function in 1891.]

Problem 19. A set $X$ is countable provided that it is finite or that there is a function $f : \mathbb{N} \to X$ which is one-to-one and onto. Show that $\mathbb{N}$ is countable, $\mathbb{Z}$ is countable, $\mathbb{Z}^n$ is countable, and $\mathbb{Q}$ is countable. Show that if $X$ is countable and $A \subset X$, then $A$ is countable.

Problem 20. Show that $[0, 1]$ is uncountable. Show that the Cantor Set $C$ is uncountable.

Problem 21. Show that if $X$ is any set, then there is no function $f : X \to 2^X$ such that $f$ is onto.

Problem 22. Suppose that $X$ and $Y$ are metric spaces and that $A \subset X$ is compact. Suppose that $f : X \to Y$ is continuous. Show that $f(A)$ is compact.

4. Connectedness

Problem 23. Suppose that $X$ is a metric space and that $A \subset X$. We say that $A$ is disconnected provided that $A = B_1 \cup B_2$ such that for every sequence $\{x_n\}_{n=1}^\infty \subset B_1$ with $x_n \to z$ as $n \to \infty$, then $z \in B_i$ for $i = 1, 2$. We say that $\{B_1, B_2\}$ is a separation for $A$. If $A$ is not disconnected, then we say that $A$ is connected. Suppose that $X$ and $Y$ are metric spaces and that $f : X \to Y$ is continuous. Show that if $A$ is connected in $X$, then $f(A)$ is connected in $Y$.

Problem 24. Show that $A \subset \mathbb{R}$ is connected if and only if $A$ is an interval.

Problem 25. State Sharkovsky’s Theorem. Prove that if $f : [a, b] \to [a, b]$ is continuous and $x_0$ is a point having period 3, then for every $n \in \mathbb{N}$, there is a point $x_n \in [a, b]$ such that $x_n$ has period $n$. 
Problem 26. Suppose $f : [a, b] \rightarrow [a, b]$ is continuous and $x_0$ is a point having period 3. How many points of period 5 are there? How many orbits of period 5? How many points of period 29 are there? How many orbits of period 29? How many points of period 229? Note that 29 and 229 are prime numbers. How many points of period 6?

Problem 27. Suppose $f : [a, b] \rightarrow [a, b]$ is continuous and $x_0$ is a point having period 5. Suppose that \{ $x_0, f(x_0) = x_1, f(x_1) = x_2, f(x_2) = x_3, f(x_3) = x_4, f(x_4) = x_0$ \} is the orbit with $x_2 < x_3 < x_4 < x_0 < x_1$. What is the Markov Graph for this orbit? What is the Adjacency Matrix for this orbit? Is there an orbit of period 3? How many orbits of period 3 are there? How many orbits of period 229?