Problem 1. Consider a recurring experiment such that the outcome each time is independent of the previous times that the experiment was performed. Suppose that the probability of a success each time the experiment is performed is \( p \), \( 0 < p < 1 \). What is the probability of ten successes in 20 experiments? What is this value for \( p = \frac{1}{4} \)? Use the simulation program to do 100 simulations with \( p = \frac{1}{4} \) and \( n = 20 \). Record the average number of successes in the 100 simulations.

Problem 2. Use the program dice to simulate rolling a die fifty times. Simulate tossing a coin fifty times using the program coin.

Problem 3. Simulate rolling ten dice using the dice program. Do this twenty times, compute the sum of the dice each time, and record the results.

Problem 4. Assume a queueing system with Poisson arrival rate of \( \alpha \) and a single server with an exponential service rate \( \sigma \). Assume that \( \sigma > \alpha > 0 \). This is an \( M/M/1/FIFO \) queue. Determine the steady-state probabilities for \( n, \{\bar{p}_n\}^\infty_{n=0} \) for this system. Determine the expected number of customers in the system, \( \mathbb{E}[n] = \pi = \sum^\infty_{n=0} n\bar{p}_n \). The solutions are \( \{\bar{p}_n = \left(\frac{\alpha}{\sigma}\right)^n \cdot \left(1 - \left(\frac{\alpha}{\sigma}\right)\right)\}^\infty_{n=0} \) and \( \mathbb{E}[n] = \frac{\left(\frac{\alpha}{\sigma}\right)}{\left(1 - \left(\frac{\alpha}{\sigma}\right)\right)} \).

Problem 5. Use the Queue program to simulate a queueing system for \( M/M/1/FIFO \) with \( \alpha = 9 \) and \( \sigma = 10 \). Simulate a queueing system for \( M/M/2/FIFO \) with \( \alpha = 9 \) and \( \sigma = 10 \). How do the results compare with the theoretical calculations for \( \{\bar{p}_n\}^\infty_{n=0} \) in each of these cases?

Problem 6. Suppose that points are distributed in an interval \([0, t]\) as a Poisson process with rate \( \lambda > 0 \). Show that the probability of the number of points in the interval being \( k \) is given by the following Poisson Distribution.

\[
\frac{(\lambda \cdot t)^k}{k!} \exp(-\lambda t)
\]

Problem 7. Assume that you have a program that will generate a sequence of independent random numbers from the uniform distribution on \([0, 1]\). Your calculator has a program that is purported to have this property. It is the \texttt{rand()} function. Determine a program that will generate independent random numbers from the exponential waiting time with parameter \( \alpha \). The probability density function for this waiting time is given by \( f(t) = \alpha \cdot e^{-\alpha \cdot t} \) and the cumulative distribution function is given by \( F(t) = 1 - e^{-\alpha \cdot t} \).
Problem 8. Suppose that there are an infinite number of servers in the queueing system $M/M/\infty$. Suppose that the arrival rate is $\alpha$ and the service rate for each server is $\sigma$. Determine the steady-state probabilities $\{p_n\}_{n=0}^{\infty}$ for this system. Explain how this could be used to model the population of erythrocytes in human blood. What would $\alpha$ and $\sigma$ be in this case? Determine approximate numerical values for $\alpha$ and $\sigma$ in this case.

Problem 9. In Gambler’s Ruin two players engage in a game of chance in which A wins a dollar from B with probability $p$ and B wins a dollar from A with probability $q = 1 - p$. There are $N$ dollars between A and B and A begins the $n$ dollars. They continue to play the game until A or B has won all of the money. What is the probability that A will end up with all the money assuming that $p > q$. Assume that $p = \frac{20}{38}$ which happens to be the house advantage in roulette. What is the probability that A will win all the money if $n = $100 and $N = $1,000,000,000? Assume that $p = \frac{20}{38}$ which happens to be the house advantage in roulette. What is the probability that A will win all the money if $n = $10 and $N = $100? Estimate this by Monte-Carlo simulation using the `gamblerruin` program in your calculator library.

Problem 10. Suppose that Urn I is chosen with probability $\frac{1}{2}$ and Urn II is also chosen with probability $\frac{1}{2}$. Suppose that Urn I has 5 white balls and 7 black balls and Urn II has 8 white balls and 3 black balls. After one of the urns is chosen, a ball is chosen at random from the urn. What is the probability that the urn was Urn I given that the ball chosen was white?

Problem 11. A test for a disease is positive with probability .95 when administered to a person with the disease. It is positive with probability .03 when administered to a person not having the disease. Suppose that the disease occurs in one in a million persons. Suppose that the test is administered to a person at random and the test is positive. What is the probability that the person has the disease. Solve this exactly using Bayes’ Theorem. Estimate the probability by Monte-Carlo simulation using the program `medicaltest` in your calculator library.

Problem 12. Let $f : [a, b] \to [a, b]$ be continuous. Show that $\frac{1}{n} \cdot \sum_{i=1}^{n} f((b-a) \cdot \text{rand}() + a) \cdot (b-a)$ converges to $\int_{a}^{b} f(x)dx$ as $n \to \infty$. This limit is the basic underlying principle of Monte-Carlo simulation.