# MAD 4401 QUIZ 1 SPRING 2018 - JAMES KEESLING

**Problem 1.** Consider a recurring experiment such that the outcome each time is independent of the previous times that the experiment was performed. Suppose that the probability of a *success* each time the experiment is performed is $p$, $0 < p < 1$. What is the probability of ten successes in 20 experiments? What is this value for $p = \frac{1}{4}$? Use the **simulation** program to do 100 simulations with $p = \frac{1}{4}$ and $n = 20$. Record the average number of successes in the 100 simulations.

**Problem 2.** Use the program **dice** to simulate rolling a die fifty times. Simulate tossing a coin fifty times using the program **coin**.

**Problem 3.** Simulate rolling ten dice using the **dice** program. Do this twenty times, compute the sum of the dice each time, and record the results.

**Problem 4.** Assume a queueing system with Poisson arrival rate of $\alpha$ and a single server with an exponential service rate $\sigma$. Assume that $\sigma > \alpha > 0$. This is an $M/M/1/FIFO$ queue. Determine the steady-state probabilities for $n$, $\{\overline{p}_n\}_{n=0}^{\infty}$ for this system. Determine the expected number of customers in the system, $\mathbb{E}[n] = \overline{n} = \sum_{n=0}^{\infty} n\overline{p}_n$. The solutions are $\left\{\overline{p}_n = \left(\frac{\alpha}{\sigma}\right)^n \cdot \left(1 - \left(\frac{\alpha}{\sigma}\right)\right)\right\}_{n=0}^{\infty}$ and $\mathbb{E}[n] = \frac{\left(\frac{\alpha}{\sigma}\right)}{\left(1-\left(\frac{\alpha}{\sigma}\right)\right)}$.

**Problem 5.** Use the **Queue** program to simulate a queueing system for $M/M/1/FIFO$ with $\alpha = 9$ and $\sigma = 10$. Simulate a queueing system for $M/M/2/FIFO$ with $\alpha = 9$ and $\sigma = 10$. How do the results compare with the theoretical calculations for $\{\overline{p}_n\}_{n=0}^{\infty}$ in each of these cases?

**Problem 6.** Suppose that points are distributed in an interval $[0, t]$ as a Poisson process with rate $\lambda > 0$. Show that the probability of the number of points in the interval being $k$ is given by the following **Poisson Distribution**.

$$\frac{(\lambda \cdot t)^k}{k!} \exp(-\lambda t)$$

**Problem 7.** Assume that you have a program that will generate a sequence of independent random numbers from the uniform distribution on $[0, 1]$. Your calculator has a program that is purported to have this property. It is the **rand()** function. Determine a program that will generate independent random numbers from the exponential waiting time with parameter $\alpha$. The probability density function for this waiting time is given by $f(t) = \alpha \cdot e^{-\alpha \cdot t}$ and the cumulative distribution function is given by $F(t) = 1 - e^{-\alpha \cdot t}$.

**Problem 8.** Suppose that there are an infinite number of servers in the queueing system $M/M/\infty$. Suppose that the arrival rate is $\alpha$ and the service rate for each server is $\sigma$. Determine the steady-state probabilities $\{\overline{p}_n\}_{n=0}^{\infty}$ for this system. Explain how this could be used to model the population of erythrocytes in human blood. What would $\alpha$ and $\sigma$ be in this case? Determine approximate numerical values for $\alpha$ and $\sigma$ in this case.

**Problem 9.** In **Gambler's Ruin** two players engage in a game of chance in which A wins a dollar from B with probability $p$ and B wins a dollar from A with probability $q = 1 - p$. There are $N$ dollars between A and B and A begins the $n$ dollars. They continue to play the game until A or B has won all of the money. What is the probability that A will end up with all the money assuming that $p > q$. Assume that $p = \frac{20}{38}$ which happens to be the house advantage in roulette. What is the probability that A will win all the money if $n = \$100$ and $N = \$1,000,000,000$? Assume that $p = \frac{20}{38}$ which happens to be the house advantage in roulette. What is the probability that A will win all the money if $n = \$10$ and $N = \$100$? Estimate this by Monte-Carlo simulation using the **gamblerruin** program in your calculator library.

**Problem 10.** Suppose that Urn I is chosen with probability $\frac{1}{2}$ and Urn II is also chosen with probability $\frac{1}{2}$. Suppose that Urn I has 5 white balls and 7 black balls and Urn II has 8 white balls and 3 black balls. After one of the urns is chosen, a ball is chosen at random from the urn. What is the probability that the urn was Urn I given that the ball chosen was white?

**Problem 11.** A test for a disease is positive with probability .95 when administered to a person with the disease. It is positive with probability .03 when administered to a person not having the disease. Suppose that the disease occurs in one in a million persons. Suppose that the test is administered to a person at random and the test is positive. What is the probability that the person has the disease. Solve this exactly using Bayes' Theorem. Estimate the probability by Monte-Carlo simulation using the program **medicaltest** in your calculator library.

**Problem 12.** Let $f : [a, b] \to [a, b]$ be continuous. Show that $\frac{1}{n} \cdot \sum_{i=1}^{n} f((b-a) \cdot \mathbf{rand()} + a) \cdot (b - a)$ converges to $\int_a^b f(x)dx$ as $n \to \infty$. This limit is the basic underlying principle of **Monte-Carlo simulation**.