MAD 4401 QUIZ 3 FALL 2017 - JAMES KEESLING

Problem 1. Determine the coefficients to compute the first derivative of $f(x) = \sin(x^2)$ at $a = 2$ using the points $\{a-2h, a-h, a, a+h, a+2h\}$. Give the estimate of the derivative as a function of h. Determine the best value of h for the greatest accuracy of the answer. How many digits accuracy can you expect with this choice of h?

Problem 2. Determine the coefficients to compute the second and third derivative of $f(x) = \sin(x^2)$ at $a = 2$ using the points $\{a-2h, a-h, a, a+h, a+2h\}$. Give the estimate of the second and third derivatives as functions of h. Determine the best value of h for the greatest accuracy of the answer. How many digits accuracy can you expect with this choice of h?

Problem 3. Suppose that $k \leq n$. Show that when estimating the kth derivative of $f(x)$ at a using the points $\{a + m_0 \cdot h, a + m_1 \cdot h, a + m_2 \cdot h, \cdots, 1 + m_n \cdot h\}$, the result is exact for $f(x)$ a polynomial of degree $p \leq n$.

Problem 4. Estimate $\frac{d^n f}{dx^n}$ at $x = a$ using the points $\{a - 4 \cdot h, a - 2 \cdot h, a - h, a, a + h, a + 2 \cdot h, a + 4 \cdot h\}$. For which n can this be done? What is the best h? What is the error?

Problem 5. Solve the differential equation for $\frac{dx}{dt} = f(t, x) = t \cdot x^2$ with $x(0) = 1$. Solve using Picard iteration for five iterations. Solve using the Taylor method of order 3,4, and 5. Solve using the Euler method, modified Euler, Heun, and Runge-Kutta methods using $h = \frac{1}{20}$ and $n = 20$. Compare the answers and the errors for each of these methods.

Problem 6. How would you go about solving the differential equation $\frac{d^2x}{dt^2} = -x$ with $x(0) = 1$ and $x'(0) = 1$ with each of the methods listed in the previous problem. What changes would need to be made in the programs? Solve this problem as a linear differential equation using the **linearode** program. Solve on the interval $[0, 1]$ with $h = \frac{1}{10}$.

Problem 7. Find a Taylor expansion for the solution $x(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ for the differential equation $\frac{dx}{dt} = t \cdot x$ with the boundary condition $x(0) = 1$. Solve for $\{a_0, a_1, a_2, a_3, a_4, a_5\}$. Do this by hand solving for these coefficients recursively. Solve for the coefficients using the Taylor Method program included in your program collection. Can you determine the general a_n?

Problem 8. Consider the following differential equation.

$$\frac{dx}{dt} = t \cdot x$$
$$x(0) = 1$$

Solve on the interval $[0,1]$ using $h = .1$. Solve using the Taylor Method of degree 4, 5, 6, 7, and 8. Compare these results with Runge-Kutta using the same h.

Problem 9. Compare Euler, Heun, and Runge-Kutta on $[0,1]$ using $h = .1$.

$$\frac{dx}{dt} = t \cdot x$$
$$x(0) = 1$$

Problem 10. Use the Euler method to solve the following differential equation

$$\frac{dx}{dt} = x$$
$$x(0) = 1$$

Solve on $[0,1]$ using $h = \frac{1}{n}$. Do this by hand to show that $x_i = \left(1 + \frac{1}{n}\right)^i$. What does this say about the following limit?

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Problem 11. Solve $\frac{dx}{dt} = M \cdot x$ with $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Problem 12. Solve $\frac{dx}{dt} = M \cdot x$ with $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and with $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Problem 13. Convert $\frac{d^2x}{dt^2} + x = 0$ to a first-order differential equation. Solve over the interval $[0, \pi]$ with $h = \frac{\pi}{10}$ assuming the initial conditions $x(0) = 1$ and $x'(0) = 0$. Use the program **linearode**.

Problem 14. Convert $\frac{d^3x}{dt^3} + x = 0$ to a first-order differential equation. Solve this equation over the interval $[0,1]$ for the initial conditions $x''(0) = 0$, $x'(0) = 1$, and $x(0) = 0$. Use the program **linearode**.

Problem 15. Explain the basis for the **bowling** program. Run some examples with different values for the probability of a strike, spare, and open frame for each frame. Discuss the results.

Problem 16. Consider a recurring experiment such that the outcome each time is independent of the previous times that the experiment was performed. Suppose that the probability of a *success* each time the experiment is performed is p, $0 < p < 1$. What is the probability of ten successes in 20 experiments? What is this value for $p = \frac{1}{4}$? Use the **simulation** program to do 100 simulations with $p = \frac{1}{4}$ and $n = 20$. Record the average number of successes in the 100 simulations.

Problem 17. Use the program **dice** to simulate rolling a die fifty times. Simulate tossing a coin fifty times using the program **coin**.

Problem 18. Simulate rolling ten dice using the **dice** program. Do this twenty times, compute the sum of the dice each time, and record the results.

Problem 19. Assume a queueing system with Poisson arrival rate of α and a single server with an exponential service rate σ. Assume that $\sigma > \alpha > 0$. This is an $M/M/1/FIFO$ queue. Determine the steady-state probabilities for n, $\{\overline{p}_n\}_{n=0}^{\infty}$ for this system. Determine the expected number of customers in the system, $\mathbb{E}[n] = \overline{n} = \sum_{n=0}^{\infty} n\overline{p}_n$. The solutions are $\left\{\overline{p}_n = \left(\frac{\alpha}{\sigma}\right)^n \cdot \left(1 - \left(\frac{\alpha}{\sigma}\right)\right)\right\}_{n=0}^{\infty}$ and $\mathbb{E}[n] = \frac{\left(\frac{\alpha}{\sigma}\right)}{\left(1-\left(\frac{\alpha}{\sigma}\right)\right)}$.

Problem 20. Use the **Queue** program to simulate a queueing system for $M/M/1/FIFO$ with $\alpha = 9$ and $\sigma = 10$. Simulate a queueing system for $M/M/2/FIFO$ with $\alpha = 9$ and $\sigma = 10$. How do the results compare with the theoretical calculations for $\{\overline{p}_n\}_{n=0}^{\infty}$ in each of these cases?

Problem 21. Suppose that points are distributed in an interval $[0, t]$ as a Poisson process with rate $\lambda > 0$. Show that the probability of the number of points in the interval being k is given by the following **Poisson Distribution**.

$$\frac{(\lambda \cdot t)^k}{k!} \exp(-\lambda t)$$

Problem 22. Assume that you have a program that will generate a sequence of independent random numbers from the uniform distribution on $[0, 1]$. Your calculator has a program that is purported to have this property. It is the **rand()** function. Determine a program that will generate independent random numbers from the exponential waiting time with parameter α. The probability density function for this waiting time is given by $f(t) = \alpha \cdot e^{-\alpha \cdot t}$ and the cumulative distribution function is given by $F(t) = 1 - e^{-\alpha \cdot t}$.

Problem 23. Suppose that there are an infinite number of servers in the queueing system $M/M/\infty$. Suppose that the arrival rate is α and the service rate for each server is σ. Determine the steady-state probabilities $\{\overline{p}_n\}_{n=0}^{\infty}$ for this system. Explain how this could be used to model the population of erythrocytes in human blood. What would α and σ be in this case? Determine approximate numerical values for α and σ in this case.

Problem 24. In **Gambler's Ruin** two players engage in a game of chance in which A wins a dollar from B with probability p and B wins a dollar from A with probability $q = 1 - p$. There are N dollars between A and B and A begins the n dollars. They continue to play the game until A or B has won all of the money. What is the probability that A will end up with all the money assuming that $p > q$. Assume that $p = \frac{20}{38}$ which happens to be the house advantage in roulette. What is the probability that A will win all the money if $n = \$100$ and $N = \$1,000,000,000$? Assume that $p = \frac{20}{38}$ which happens to be the house advantage in roulette. What is the probability that A will win all the money if $n = \$10$ and $N = \$100$? Estimate this by Monte-Carlo simulation using the **gamblerruin** program in your calculator library.

Problem 25. Suppose that Urn I is chosen with probability $\frac{1}{2}$ and Urn II is also chosen with probability $\frac{1}{2}$. Suppose that Urn I has 5 white balls and 7 black balls and Urn II has 8 white balls and 3 black balls. After one of the urns is chosen, a ball is chosen at random from the urn. What is the probability that the urn was Urn I given that the ball chosen was white?

Problem 26. A test for a disease is positive with probability .95 when administered to a person with the disease. It is positive with probability .03 when administered to a person not having the disease. Suppose that the disease occurs in one in a million persons. Suppose that the test is administered to a person at random and the test is positive. What is the probability that the person has the disease. Solve this exactly using Bayes' Theorem. Estimate the probability by Monte-Carlo simulation using the program **medicaltest** in your calculator library.

Problem 27. Let $f : [a, b] \to [a, b]$ be continuous. Show that $\frac{1}{n} \cdot \sum_{i=1}^{n} f((b-a) \cdot \mathbf{rand()} + a) \cdot (b - a)$ converges to $\int_a^b f(x)dx$ as $n \to \infty$. This limit is the basic underlying principle of **Monte-Carlo simulation**.