REVIEW MAD 4401 TEST 2 FALL 2018 - JAMES KEESLING

Do all problems and show all work. Each problem is worth 20 points. Partial credit will be given for correct reasoning when the final answer may be incorrect. Credit will be deducted if reasoning is wrong even if the final answer is correct.

1. Differential Equations

Problem 1. Explain **Picard Iteration** as a method to solve a differential equation.

$$\frac{dx}{dt} = f(x,t)$$
$$x(0) = x_0$$

Problem 2. Use Picard Iteration using the program **picard** to approximate the solution of the the differential equation. Use Taylor Method via the **taylormethod** program to approximate the solution with a polynomial of degree 10.

$$\frac{dx}{dt} = \sin(t) \cdot x$$
$$x(0) = 1$$

Problem 3. Determine a first order system of differential equations for the following differential equation.

$$\frac{d^2x}{dt^2} + x = 0$$
$$x(0) = 1,\ x'(0) = 0$$

Solve using the program **linearode**. Approximate the solution using **picard2**.

Problem 4. Determine a first order system of differential equations for the following differential equation.

$$\frac{d^3x}{dt^3} + x = 0$$
$$x(0) = 0,\ x'(0) = 1,\ x''(0) = 0$$

Solve the resulting equation using the program **linearode**. Solve using the program **picard2**.

Problem 5. Numerically solve the following differential equation using the Euler Method, the Heun Method, and using Runge Kutta. Use $h = \frac{1}{10}$ and $n = 10$ for each method. Compare the results and determine the likely error in each result.

$$\frac{dx}{dt} = t^2 \cdot x$$
$$x(0) = 1$$

Problem 6. Use Picard Iteration using **picard2** to determine the third iteration of the following system of differential equations.

$$\frac{dx}{dt} = t \cdot y \quad x(0) = 1$$
$$\frac{dy}{dt} = \sin(t) \cdot z \quad y(0) = -1$$
$$\frac{dz}{dt} = \exp(t) \cdot x \quad z(0) = 0$$

2. Stochastic Simulation and Queueing Theory

Problem 7. Explain the basis for the **bowling** program. Run some examples with different values for the probability of a strike, spare, and open frame for each frame. Discuss the results.

Problem 8. Consider a recurring experiment such that the outcome each time is independent of the previous times that the experiment was performed. Suppose that the probability of a *success* each time the experiment is performed is p, $0 < p < 1$. What is the probability of ten successes in 20 experiments? What is this value for $p = \frac{1}{4}$? Use the **simulation** program to do 100 simulations with $p = \frac{1}{4}$ and $n = 20$. Record the average number of successes in the 100 simulations.

Problem 9. Use the program **dice** to simulate rolling a die fifty times. Simulate tossing a coin fifty times using the program **coin**.

Problem 10. Simulate rolling ten dice using the **dice** program. Do this twenty times, compute the sum of the dice each time, and record the results.

Problem 11. Assume a queueing system with Poisson arrival rate of α and a single server with an exponential service rate σ. Assume that $\sigma > \alpha > 0$. This is an $M/M/1/FIFO$

queue. Determine the steady-state probabilities for n, $\{\overline{p}_n\}_{n=0}^{\infty}$ for this system. Determine the expected number of customers in the system, $\mathbb{E}[n] = \overline{n} = \sum_{n=0}^{\infty} n\overline{p}_n$. The solutions are $\left\{\overline{p}_n = \left(\frac{\alpha}{\sigma}\right)^n \cdot \left(1 - \left(\frac{\alpha}{\sigma}\right)\right)\right\}_{n=0}^{\infty}$ and $\mathbb{E}[n] = \frac{\left(\frac{\alpha}{\sigma}\right)}{\left(1-\left(\frac{\alpha}{\sigma}\right)\right)}$.

Problem 12. Use the **Queue** program to simulate a queueing system for $M/M/1/FIFO$ with $\alpha = 9$ and $\sigma = 10$. Simulate a queueing system for $M/M/2/FIFO$ with $\alpha = 9$ and $\sigma = 10$. How do the results compare with the theoretical calculations for $\{\overline{p}_n\}_{n=0}^{\infty}$ in each of these cases?

Problem 13. Suppose that points are distributed in an interval $[0, t]$ as a Poisson process with rate $\lambda > 0$. Show that the probability of the number of points in the interval being k is given by the following **Poisson Distribution**.

$$\frac{(\lambda \cdot t)^k}{k!} \exp(-\lambda t)$$

Problem 14. Assume that you have a program that will generate a sequence of independent random numbers from the uniform distribution on $[0, 1]$. Your calculator has a program that is purported to have this property. It is the **rand()** function. Determine a formula for your calculator that will generate independent random numbers from the exponential waiting time with parameter α. The probability density function for this waiting time is given by $f(t) = \alpha \cdot e^{-\alpha \cdot t}$ and the cumulative distribution function is given by $F(t) = 1 - e^{-\alpha \cdot t}$.

Problem 15. Suppose that there are an infinite number of servers in the queueing system $M/M/\infty$. Suppose that the arrival rate is α and the service rate for each server is σ. Determine the steady-state probabilities $\{\overline{p}_n\}_{n=0}^{\infty}$ for this system. Explain how this could be used to model the population of erythrocytes in human blood. What would α and σ be in this case? Assume that there are 25 trillion erythrocytes in the human body and that each has a lifespan of 100 days. Determine approximate numerical values for α and σ in this case.

Problem 16. In **Gambler's Ruin** two players engage in a game of chance in which A wins a dollar from B with probability p and B wins a dollar from A with probability $q = 1 - p$. There are N dollars between A and B and A begins the n dollars. They continue to play the game until A or B has won all of the money. What is the probability that A will end up with all the money assuming that $p > q$. Assume that $p = \frac{20}{38}$ which happens to be the house advantage in roulette. What is the probability that A will win all the money if $n = \$100$ and $N = \$1,000,000,000$? Assume that $p = \frac{20}{38}$ which happens to be the house advantage in roulette. What is the probability that A will win all the money if

$n = \$10$ and $N = \$100$? Estimate this by Monte-Carlo simulation using the **gamblerruin** program in your calculator library.

Problem 17. Suppose that Urn I is chosen with probability $\frac{1}{2}$ and Urn II is also chosen with probability $\frac{1}{2}$. Suppose that Urn I has 5 white balls and 7 black balls and Urn II has 8 white balls and 3 black balls. After one of the urns is chosen, a ball is chosen at random from the urn. What is the probability that the urn was Urn I given that the ball chosen was white?

Problem 18. A test for a disease is positive with probability .95 when administered to a person with the disease. It is positive with probability .03 when administered to a person not having the disease. Suppose that the disease occurs in one in a million persons. Suppose that the test is administered to a person at random and the test is positive. What is the probability that the person has the disease. Solve this exactly using Bayes' Theorem. Estimate the probability by Monte-Carlo simulation using the program **medicaltest** in your calculator library.

Problem 19. Let $f : [a, b] \to [a, b]$ be continuous. Show that $\frac{1}{n} \cdot \sum_{i=1}^{n} f((b-a) \cdot \mathbf{rand()} + a) \cdot (b-a)$ converges to $\int_a^b f(x)dx$ as $n \to \infty$. This limit is the basic underlying principle of **Monte-Carlo simulation**.