MAD 2302 DIFFERENTIAL EQUATIONS PRACTICE PROBLEMS

JAMES KEESLING

The problems that follow illustrate the methods covered in class. They are typical of the types of problems that will be on the tests.

1. Introductory Material

Definition A *differential equation* is given as

$$\frac{dx}{dt} = f(t,x)$$

$$x(t_0) = x_0$$

where $f(t,x)$ is a function that is jointly continuous in t and x. A *solution* of the differential equation is a function $x(t)$ such that the derivative of x, x', has the property that

$$x'(t) \equiv f(t, x(t))$$

and

$$x(t_0) = x_0.$$

.

Problem 1. Show that $x(t) = \exp(t)$ is a solution of the differential equation

$$\frac{dx}{dt} = x$$

with

$$x(0) = 1.$$

Problem 2. Show that $x(t) = \exp\left(\frac{t^2}{2}\right)$ is a solution of the differential equation

$$\frac{dx}{dt} = t \cdot x$$

with

$$x(0) = 1.$$

Problem 3. Show that $x(t) = \frac{1}{1-x}$ is a solution of the differential equation

$$\frac{dx}{dt} = x^2$$

with

$$x(0) = 1.$$

2. Picard Iteration

Problem 4. Do three iterations of *Picard iteration* for the differential equation

$$\frac{dx}{dt} = t \cdot x$$

with

$$x(0) = 1.$$

Start with $x_0(t) \equiv 1$ and determine $x_1(t)$, $x_2(t)$, and $x_3(t)$.

3. Separation of Variables

Problem 5. Solve the differential equation.

$$\frac{dx}{dt} = \sin(t) \cdot x$$

with

$$x(0) = 1$$

4. The Euler Method and the Taylor Method

Problem 6. Give a numerical solution of the differential equation using the Euler method.

$$\frac{dx}{dt} = x$$

with

$$x(0) = 1$$

Solve over the interval $[0, 1]$ using $h = \frac{1}{10}$ and $n = 10$. How close is the numerical solution to the theoretical solution at $t = 1$? Solve using $h = \frac{1}{100}$ and $h = \frac{1}{1000}$. How is the error changing at $t = 1$?

Problem 7. Give a numerical solution of the differential equation using the Taylor method.

$$\frac{dx}{dt} = x$$

with

$$x(0) = 1$$

Solve over the interval $[0, 1]$ using $h = \frac{1}{10}$ and $n = 10$. Do this with the Taylor method of order 2, 3, and 4. How close is the numerical solution to the theoretical solution at $t = 1$? Solve using these Taylor methods for $h = \frac{1}{100}$ and $h = \frac{1}{1000}$. How is the error changing at $t = 1$?

Problem 8. Give a numerical solution of the differential equation using the Taylor method.

$$\frac{dx}{dt} = \sin(t) \cdot x$$

with

$$x(0) = 1$$

Solve over the interval $[0, 1]$ using $h = \frac{1}{10}$ and $n = 10$. Do this with the Taylor method of order 2, 3, and 4. How close is the numerical solution to the theoretical solution at $t = 1$? Solve using these Taylor methods for $h = \frac{1}{100}$ and $h = \frac{1}{1000}$. How is the error changing at $t = 1$?

5. Exact Equations

Problem 9. Express the following equation as a differential equation. Here C is a constant.

$$\frac{x^2}{9} + \frac{y^2}{25} = C$$

This will be a family of ellipses.

Determine a differential equation describing the curves such that at each point that the curves intersect one of the ellipses above, the two will be perpendicular. Solve this differential equation. This family of curves is called the *family of perpendicular curves.*

Problem 10. Express the following equation as a differential equation. Here C is a constant.

$$F(x, y) = C$$

Determine a differential equation describing the family curves which are perpendicular to the above curves.

Problem 11. Solve the following differential equations.

$$x\,dx + y\,dy = 0$$

$$(x^3 + y)\,dx + (y + x + 3)\,dy = 0$$

6. Integrating Factors

Problem 12. Solve the following differential equations.

$$x^2 dx + xy\, dy = 0$$

$$(x^3 + y) \cdot y\, dx + (y + x + 3) \cdot y\, dy = 0$$

7. Linear First-Order Differential Equations

Problem 13. Solve the following differential equations.

$$\frac{dx}{dt} + \frac{x}{t} = t\,\sin(t)$$

$$\frac{dx}{dt} + t\,x = \sin(t)$$

8. Homogeneous Equations

Problem 14. Solve the following differential equations.

$$(x\,y + y^2 + x^2)\,dx - x^2\,dy = 0$$

$$(y^2 + x^2)\,dx + x^2\,dy = 0$$

9. Some Appliations

Problem 15. Show that atmospheric pressure decreases exponentially with altitude. Show that underwater pressure increases linearly with depth.

Problem 16. What is the air pressure at 18,000 feet? What is the air pressure at 36,000 feet?

Problem 17. What is the pressure 10 feet below the surface of the ocean? What is the pressure one mile under the surface of the ocean? Assume that the density of seawater is 1029 kg/m^3?

Problem 18. Consider a tank containing 1000 liters of water. Suppose that water flows into the tank at a rate of 6 liters per minute such that the incoming water has $\frac{1}{10}$ kilogram of salt per liter. Suppose that water flows out of the tank at a rate of 6 liters per minute. Let $x(t)$ be the number of kilograms of salt in the water at time t. Assume that $x(0) = 0$. Assuming complete mixing of the incoming water with the water in the tank, determine a differential equation for the salt in the tank. Solve the resulting differential equation.

Problem 19. Consider a tank containing 1000 liters of water. Suppose that water flows into the tank at a rate of 6 liters per minute such that the incoming water has $\frac{1}{10}$ kilogram of salt per liter. Suppose that water flows out of the tank at a rate of 3 liters per minute. Let $x(t)$ be the number of kilograms of salt in the water at time t. Assume that $x(0) = 0$. Assuming complete mixing of the incoming water with the water in the tank, determine a differential equation for the salt in the tank. Solve the resulting differential equation.

Problem 20. Consider a mass attached to a spring resting on a frictionless surface as in the accompanying diagram. Assume that the spring obeys Hooke's Law, that is, $F = -k \cdot x$ where k is Hooke's constant and x is the distance that the mass is from the equilibrium position for the spring. Determine the differential equation that governs the movement of the mass. Describe the possible solutions of the equation.

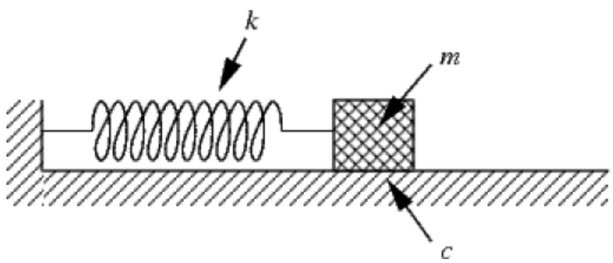

FIGURE 1. Mass-Spring Arrangement

Problem 21. Consider a mass attached to a spring resting on a frictionless surface as in the accompanying diagram. Assume that the spring obeys Hooke's Law, that is, $F = -k \cdot x$ where k is Hooke's constant and x is the distance that the mass is from the equilibrium position for the spring. Assume also that there is a viscous force due to the motion of the mass through the air. Assume that this drag force is given by $-b\frac{dx}{dt}$ with $b > 0$. Determine the differential equation that governs the movement of the mass. Describe the possible solutions of the equation.

10. Linear Homogeneous Second-Order Differential Equations with Constant Coefficients

Problem 22. Consider a homogeneous second order differential equation with constant coefficients. It can be put in the following form.

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0 \qquad (1)$$

The polynomial $p(x) = a_2 m^2 + a_1 m + a_0$ is called the *auxiliary polynomial.* Explain how the roots of $p(x)$ determine the solutions of (1). Describe the possibilities for the solution based on the roots.

Problem 23. Solve the following differential equations.

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$$
$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 17x = 0$$
$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 0$$
$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 0$$

11. Linear Non-Homogeneous Differential Equations with Constant Coefficients

Problem 24. Solve the following differential equations using the *method of undetermined coefficients.*

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 5$$
$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 17x = t$$
$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = \sin(t)$$
$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^{-t}$$
$$\frac{d^3x}{dt^3} - 2\frac{dx}{dt} + x = e^{-t}$$

12. Linear Independence and the Wronskian

Problem 25. Compute the Wronskian of the following collections of functions. Which collections are linearly independent?

$$\{\sin(t), \cos(t)\}$$
$$\{\exp(t), \exp(2t)\}$$
$$\{\sin(t), \exp(t), \exp(2t)\}$$
$$\{1, \sin^2(t), \cos^2(t)\}$$
$$\{\exp(t), 2\exp(t)\}$$
$$\{\exp(t), 2t\exp(t)\}$$

Problem 26. Determine linear differential equations that have the following sets of functions as solutions. In which cases are all linear combinations of the functions the complete solution of the differential equation?

$$\{\sin(t), \cos(t)\}$$
$$\{\exp(t), \exp(2t)\}$$
$$\{\sin(t), \exp(t), \exp(2t)\}$$
$$\{\exp(t), 2t\exp(t)\}$$

Problem 27. Solve the following differential equations using the *method of variation of parameters*. Explain the steps involved.

$$\frac{d^3x}{dt^3} - 3\frac{dx}{dt} + 2x = \exp(2t)$$
$$\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} + \frac{dx}{dt} + 2x = \exp(2t)$$

13. Matrix Methods

Problem 28. Convert the following differential equations to first order and solve using matrix methods. Assume that $x(0) = 1$, $x'(0) = -1$, and $x''(0) = 2$. Solve numerically over $[0, 1]$ using $h = 1/10$ and $n = 10$.

$$\frac{d^3x}{dt^3} - 3\frac{dx}{dt} + 2x = \exp(2t)$$
$$\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} + \frac{dx}{dt} + 2x = \exp(2t)$$

Problem 29. Solve the following differential equation using *matrix methods*.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Assume that $x_1(0) = 2$, $x_2(0) = 1$, and $x_3(0) = 3$. Give a numerical solution over $[0, 1]$ using step-size $h = \frac{1}{10}$.

14. Laplace Transforms

Problem 30. Use the definition of the *Laplace transform* to determine the Laplace transforms $\mathscr{L}\{f(t)\} = F(s)$ for the following functions.

$$f(t) = 1$$
$$f(t) = t$$
$$f(t) = t^2$$
$$f(t) = t^n$$
$$f(t) = e^{at}$$
$$f(t) = \sin(bt)$$

$$f(t) = \cos(bt)$$

Problem 31. Determine $\mathscr{L}\{f'(t)\}$ assuming that $\mathscr{L}\{f(t)\} = F(s)$.

Problem 32. Determine $\mathscr{L}\{\exp(at) \cdot f(t)\}$ assuming that $\mathscr{L}\{f(t)\} = F(s)$.

Problem 33. Assume that $\mathscr{L}\{f(t)\} = F(s)$. Determine $g(t)$ such that $\mathscr{L}\{g(t)\} = \frac{dF(s)}{ds}$.

Problem 34. Solve the following differential equations using Laplace transforms.

$$\frac{dx}{dt} = ax \qquad x(0) = 1$$
$$\frac{d^2x}{dt^2} + x = 0 \qquad x(0) = 1 \qquad x'(0) = 0$$

Problem 35. Solve the following system of differential equations using Laplace transforms.

$$\frac{dx_1}{dt} = 2x_1 + 3x_2$$
$$\frac{dx_2}{dt} = x_1 + 2x_2$$
$$x_1(0) = 2 \quad x_2(0) = 1$$

Problem 36. Solve the following system of differential equations using Laplace transforms.

$$\frac{dx_1}{dt} = 2x_1 - 3x_2 + \exp(-t)$$
$$\frac{dx_2}{dt} = x_1 + 2x_2 + \cos(3t)$$
$$x_1(0) = -1 \quad x_2(0) = 3$$

Problem 37. Determine the Laplace transforms of $x_1(t)$, $x_2(t)$, and $x_3(t)$ using *matrix methods.* You need not determine the inverse of the transforms.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Assume that $x_1(0) = 2$, $x_2(0) = 1$, and $x_3(0) = 3$.