Baire Category Theorems

Definition. A *Baire space* is a topological space *X* that has the property that if $\{O_i\}_{i=1}^{\infty}$ is a sequence of dense open sets, then $\bigcap_{i=1}^{n} O_i$ is dense in *X*.

The following theorems hold.

Theorem 1. If *X* is a complete metric space, then *X* is a Baire space.

Theorem 2. If *X* is a compact Hausdorff space, then *X* is a Baire space.

Theorem 3. If *X* is a Baire space and *A* is a dense G_δ in *X*, then *A* is also a Baire space.

Theorem 4. The set of irrationals with the usual metric forms a Baire space.

Definition. A completely regular space is an *absolute* G_δ if it is a G_δ in every space in which it is embedded as a dense subset.

Theorem 5. A completely regular space *X* is an absolute G_δ if and only if *X* is a G_δ in its Stone-Cech compactification.

Theorem 6. Every complete separable metric space *X* is an absolute G_δ.

Proof of Theorem 1.

Suppose that $\{O_i\}_{i=1}^{\infty}$ is a sequence of dense open sets in *X*. We need to show that $\bigcap_{i=1}^{n} O_i$ is dense in *X*. Let $x \in X$ and $\varepsilon > 0$. We need to show that there is a $z \in \bigcap_{i=1}^{n} O_i \cap B_\varepsilon(x)$. We will do this by finding a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ converging to the required *z*.

Let $\delta_1 = \frac{\varepsilon}{3}$ and let x_1 be any point in $B_{\delta_1}(x) \cap O_1$. Such a point exists because O_1 is dense in *X*.

Let $0 < \delta_2 \leq \frac{\varepsilon}{9}$ such that $\overline{B_{\delta_2}(x_1)} \subset O_1 \cap B_{\delta_1}(x)$. Such a δ_2 exists because $O_1 \cap B_{\delta_1}(x)$ is open. Let x_2 be any point in $B_{\delta_2}(x_1) \cap O_2$. The point x_2 exists because O_2 is dense.

Let x_3 be a point in $B_{\delta_2}(x_2) \cap O_3$ and let $0 < \delta_3 \le \dfrac{\varepsilon}{3^3}$ be such that $\overline{B_{\delta_3}(x_3)} \subset B_{\delta_2}(x_2) \cap O_3$. The point x_3 exists because O_3 is dense. The number δ_3 exists because $B_{\delta_2}(x_2) \cap O_3$ is open.

We continue inductively using the above argument. Suppose that a sequence of points $\{x_1, x_2, \ldots, x_n\}$ has been chosen along with a sequence of positive numbers $\{\delta_1, \delta_2, \ldots, \delta_n\}$ having the following properties.

(1) $B_{\delta_i}(x_i) \subset O_i$ for $1 \le i \le n$,

(2) $0 < \delta_i \le \dfrac{\varepsilon}{3^i}$ for $1 \le i \le n$,

(3) $x_{i+1} \in B_{\delta_i}(x_i) \cap O_{i+1}$ for $1 \le i < n$, and

(4) $\overline{B_{\delta_{i+1}}(x_{i+1})} \subset B_{\delta_i}(x_i) \cap O_{i+1}$ for $1 \le i < n$.

Then we let x_{n+1} be a point in $B_{\delta_n}(x_n) \cap O_{n+1}$ and choose $0 < \delta_{n+1} \le \dfrac{\varepsilon}{3^{n+1}}$ such that $\overline{B_{\delta_{n+1}}(x_{n+1})} \subset B_{\delta_n}(x_n) \cap O_{n+1}$. This increases the sequence of points and numbers by one more element each to $\{x_1, x_2, \ldots, x_{n+1}\}$ and $\{\delta_1, \delta_2, \ldots, \delta_{n+1}\}$. The increased sequences have the above four properties.

By induction there are infinite sequences $\{x_1, x_2, \ldots\}$ and $\{\delta_1, \delta_2, \ldots\}$ such that

(1) $B_{\delta_i}(x_i) \subset O_i$ for $1 \le i < \infty$,

(2) $0 < \delta_i \le \dfrac{\varepsilon}{3^i}$ for $1 \le i < \infty$,

(3) $x_{i+1} \in B_{\delta_i}(x_i) \cap O_{i+1}$ for $1 \le i < \infty$, and

(4) $\overline{B_{\delta_{i+1}}(x_{i+1})} \subset B_{\delta_i}(x_i) \cap O_{i+1}$ for $1 \le i < \infty$.

One can easily show using (1), (2), (3), and (4) that the sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Since X is complete, this Cauchy sequence has a limit point $z = \lim_{n \to \infty} x_n$. Since z is the limit, $z \in \overline{B_{\delta_{n+1}}(x_{n+1})}$. By property (1) above $z \in O_n$ for all n. Thus, $z \in \bigcap_{n=1}^{\infty} O_n$.

Now

$$d(x,x_n) \le d(x,x_1) + d(x_1,x_2) + \cdots + d(x_{n-1},x_n)$$

$$d(x,x_n) \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3^2} + \cdots + \frac{\varepsilon}{3^n} = \varepsilon\sum_{i=1}^{n}\frac{1}{3^i} < \varepsilon\sum_{i=1}^{\infty}\frac{1}{3^i} = \frac{\varepsilon}{2}$$

As a result, $d(x,z) < \dfrac{\varepsilon}{2}$. So, $z \in B_\varepsilon(x) \cap \bigcap_{i=1}^{\infty} O_i$. This completes the proof that $\bigcap_{i=1}^{\infty} O_i$ is dense in *X*.