PROOF THAT THE DISTANCE TO A SET IS CONTINUOUS

JAMES KEESLING

In this document we prove the following theorem.

Theorem. *Let X be a metric space with metric d. Suppose that $A \subset X$ is nonempty. Define $f(x) = d(x, A) = \inf\{d(x, y) | y \in A\}$. Then $f : X \to R$ is a continuous function.*

Proof. We show that f is continuous at each $x \in X$ by showing that if V is an open set containing $f(x)$, then there is an open set $U \subset X$ such that $x \in U$ and $f(U) \subset V$.

By definition $f(x) = d(x, A) = \inf\{d(x, y)|y \in A\}$. Suppose that $f(x) \in V$ with V open in R. Then there is an $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subset V$.

We claim that $U = B_{\frac{\varepsilon}{2}}(x)$ will be the open set such that $f(U) \subset B_\varepsilon(f(x)) \subset V$. To prove this claim, let $z \in B_{\frac{\varepsilon}{2}}(x)$ and let $y \in A$. Then we have the following inequalities from the triangle inequality.

$$d(x, y) + d(x, z) \geq d(y, z)$$

and

$$d(x, z) + d(z, y) \geq d(x, y)$$

Now take the infimum over $y \in A$ in both sides of the above two inequalities.

$$d(x, A) + d(x, z) \geq d(z, A)$$

and

$$d(x, z) + d(z, A) \geq d(x, A)$$

We get the following.

$$d(x, A) + \frac{\varepsilon}{2} \geq d(z, A) \geq d(x, A) - \frac{\varepsilon}{2}$$

Thus we have the following.

$$f(x) + \varepsilon > f(z) > f(x) - \varepsilon$$

This proves that $f(z) \in B_\varepsilon(f(x)) \subset V$ and that $f(U) \subset V$.

□