MTG 5316/4302 FINAL EXAM

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Work all problems and show all work. Each problem is worth ten points. Partial credit will be given for correct statements and reasoning. Credit will be deducted for incorrect statements and reasoning.

Problem 1. Show that for every set X, there is no function $f : X \to 2^X$ such that f is onto.

Problem 2. Let (X, d) be a metric space and let $A, B \subset X$ be nonempty closed subsets of X such that $A \cap B = \emptyset$. Show that there is a continuous function $f : X \to [0, 1]$ such that $A \subset f^{-1}(1)$ and $B \subset f^{-1}(0)$.

Problem 3. Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that if f has a point of period three. Show that f has a point of period five.

Problem 4. State the Baire Category Theorem.

Problem 5. Suppose that X is a complete metric space. A function $f : X \to X$ is said to be a *contraction mapping* provided that there is a $0 < c < 1$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq c \cdot d(x, y)$. Show that if $f : X \to X$ is a contraction mapping, then there is a unique $z \in X$ such that $f(z) = z$.

Problem 6. Suppose that X is a compact metric space. Show that X is complete.

Problem 7. Suppose that X is a compact Hausdorff space (i.e., $\mathbf{T}_2$). Show that if $A \subset X$ is compact, then A is closed in X.

Problem 8. Suppose that X is compact Hausdorff ($\mathbf{T}_2$). Show that X is normal.

Problem 9. State the **Urysohn Lemma** and the **Tietze Extension Theorem**.

Problem 10. Let X be a connected metric space. Let $x, y \in X$ be distinct points. Let $\varepsilon > 0$. Show that there is a chain of open sets $\{U_1, U_2, \ldots, U_n\}$ such that $\text{diam} U_i < \varepsilon$ for each i with $x \in U_1$ and $y \in U_n$. A *chain* of sets is a collection $\{A_1, \ldots, A_n\}$ such that $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$.