MTG 5316/4302 FALL 2018 REVIEW FOR FINAL QUIZ 2

JAMES KEESLING

Problem 1. Let U be an open subset of $\mathbb{R}^n$. Show that if U is connected, then for every $a \neq b \in U$, there is a continuous $f : [0,1] \to U$ such that f is one-to-one with $f(0) = a$ and $f(1) = b$.

Problem 2. Let X be a metric space with $A \subset X$. The **closure** of A is the set $\overline{A} = \left\{x \in X \;\middle|\; \exists \{x_i\}_{i=1}^{\infty} \subset A \ni \lim_{i \to \infty} x_i = x\right\}$. Note that for any set $A \subset X$, $\overline{A}$ is closed in X. Suppose that $A \subset X$ is connected. Show that $\overline{A}$ is connected.

Problem 3. Let X be a metric space and suppose that $A_\lambda \subset X$ is connected for all $\lambda \in \Lambda$. Suppose that there is an $x_0 \in X$ such that $x_0 \in A_\lambda$ for all λ. Show that

$$\bigcup_{\lambda \in \Lambda} A_\lambda$$

is connected.

Problem 4. Let $A \subset [0,1] \times [-1,1] \subset \mathbb{R}^2$ be defined by $A = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \;\middle|\; 0 < x \leq 1\right\} \cup \{0\} \times [-1,1]$. This is the **Topologist's Sine Curve**. Show that A is connected. Show that there is no continuous function $f : [0,1] \to A$ such that $f(0) = (1, \sin(1))$ and $f(1) = (0,0)$.

Problem 5. Let X be a metric space. Define **compactness** of X in terms of open covers of X. Show that X is compact if and only if for every collection of closed sets, $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}$ if $\mathcal{F}$ has the finite intersection property, then

$$\bigcap_{\alpha \in A} F_\alpha \neq \emptyset.$$

Problem 6. Let X be a metric space. Show that X is sequentially compact if and only if X is compact.

Problem 7. Let X be a compact metric space. Suppose that $f : X \to Y$ be continuous, one-to-one, and onto. Show that $f^{-1} : Y \to X$ is also continuous. In case f and f^{-1} are both continuous, f is said to be a **homeomorphism** between X and Y and X and Y are said to be **homeomorphic**.

Problem 8. Let X be a metric space. Suppose that $\{A_i\}_{i=1}^{\infty}$ is a countable collection of compact connected sets in X such that $A_{i+1} \subset A_i$ for all i. Show that

$$\bigcap_{i=1}^{\infty} A_i$$

is compact and connected.

Problem 9. Show that the **dyadic solenoid** is compact and connected.